



MANIFOLD MATHEMATICS

Permutations & Combinations

*From ELI5 to Abstraction:
The Theory and Craft of Counting*

In Two Parts

PART I • The Grammar of Counting — Foundations to JEE Advanced
PART II • Structures and Symmetry — ISI, CMI, IAT, CSIR–NET, NBHM, Olympiad

For EAMCET • JEE Main & Advanced • IISER IAT • ISI Kolkata • CMI
IISc • CSIR–NET • NBHM • Mathematical Olympiads

MANIFOLD MATHEMATICS MONOGRAPH SERIES

Preface

Counting is the oldest branch of mathematics and, deceptively, one of the hardest. A child counts pebbles; a research mathematician counts orbits of a group action on the set of colourings of a molecular graph. Between these two acts lies a single, continuous intellectual staircase, and the purpose of this monograph is to walk the reader up every step of it — slowly at the bottom, briskly in the middle, and with full rigour at the top.

The book is written for the Indian competitive-examination ecosystem in its entire vertical extent. A student preparing for EAMCET needs fluency with $\binom{n}{r}$ manipulations under time pressure; a JEE Advanced aspirant needs derangements, multinomials and grid paths; an ISI or CMI entrance candidate needs bijective proofs and generating functions; a CSIR–NET or NBHM examinee needs group actions, Burnside’s lemma, and Pólya theory; an olympiad contestant needs all of it, plus the taste to know which tool fits which lock. Rather than write four thin pamphlets, we have written one thick book, ordered so that each audience knows exactly where its syllabus ends and its curiosity may begin.

Two recurring marginal institutions define the house pedagogy. The **ELI5 boxes** explain each new idea as one would to a bright twelve-year-old: no symbols, only pictures and stories. The **Abstract Viewpoint boxes** do the opposite: they reveal the same idea as a modern mathematician holds it — functorial, structural, general. The main text runs between these two rails. Our conviction, inherited from George F. Simmons, is that a student who can travel freely between the nursery picture and the categorical picture *owns* the concept; one who can do only the middle-level symbol manipulation merely rents it.

Theory here is intense by design. Every counting formula is proved, most twice — once by algebra, once by bijection — because in combinatorics the bijective proof is the real explanation and the algebraic proof is merely the receipt. Worked examples are drawn from and calibrated to the named examinations. Exercises are graded with level tags so that readers can filter by target.

Part I (Chapters 1–10) builds the classical grammar: the two fundamental principles, permutations in all their linear, circular and multiset variants, binomial coefficients and their identity calculus, distributions and stars-and-bars, inclusion–exclusion and derangements, factorial arithmetic, and lattice paths. Part II (Chapters 11–20) rebuilds the same subject as structure: the twelvefold way, generating functions ordinary and exponential, recurrences, Möbius inversion, pigeonhole and double counting as proof techniques, group actions with Burnside and Pólya, Catalan combinatorics, integer partitions, q -analogues, and a closing glimpse of the probabilistic method. A reader who finishes both parts will find that the word “counting” has quietly changed meaning.

Visakhapatnam, 2026

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Part I

The Grammar of Counting

Foundations to JEE Advanced

What It Means to Count

ELI5 • Counting is matching

Imagine a shepherd who cannot count past three but owns two hundred sheep. Every morning, as each sheep leaves the pen, he drops one pebble into a bag. Every evening, as each sheep returns, he removes one pebble. If the bag empties exactly when the last sheep enters, all is well. The shepherd never “counted” anything — he *matched* sheep with pebbles, one for one. That matching is the whole secret of counting. Numbers are just standardized bags of pebbles.

1.1 Counting as bijection

Modern mathematics takes the shepherd’s insight and makes it the *definition* of counting. Write $[n] = \{1, 2, \dots, n\}$ for the standard n -element set, with the convention $[0] = \emptyset$.

Definition 1.1 (Cardinality). A set A is *finite with cardinality n* , written $|A| = n$, if there exists a bijection $f: A \rightarrow [n]$. A bijection is a function that is both injective (distinct inputs give distinct outputs) and surjective (every element of the codomain is hit).

This definition immediately raises a question that a careless author would skip: could a set have two different cardinalities? Could there be bijections $A \rightarrow [m]$ and $A \rightarrow [n]$ with $m \neq n$? The answer is no, and the proof is the first genuine theorem of combinatorics.

Theorem 1.2 (Well-definedness of cardinality). *If $m \neq n$ then there is no bijection $[m] \rightarrow [n]$. Consequently every finite set has exactly one cardinality.*

Proof. We prove by induction on m the statement: for all $n \neq m$, no bijection $[m] \rightarrow [n]$ exists. For $m = 0$: a bijection $\emptyset \rightarrow [n]$ with $n \geq 1$ would have to be surjective, but the empty function hits nothing; contradiction. Assume the claim for $m - 1$ and suppose $f: [m] \rightarrow [n]$ is a bijection with $n \neq m$; since $[m] \neq \emptyset$, also $n \geq 1$. Let $k = f(m)$. Define $g: [m - 1] \rightarrow [n - 1]$ by

$$g(i) = \begin{cases} f(i) & \text{if } f(i) < k, \\ f(i) - 1 & \text{if } f(i) > k. \end{cases}$$

One checks directly that g is a bijection (it is f restricted to $[m - 1]$, followed by the order-preserving bijection $[n] \setminus \{k\} \rightarrow [n - 1]$). But $m - 1 \neq n - 1$, contradicting the induction hypothesis. $\square$

The reader should not regard Theorem 1.2 as pedantry. It is the theoretical licence for the single most powerful technique in this book:

The Bijection Principle

If there is a bijection between A and B , then $|A| = |B|$. To count a difficult set, find a bijection to an easy set.

Almost every deep identity in Part I and Part II — the symmetry $\binom{n}{r} = \binom{n}{n-r}$, the stars-and-bars formula, the Catalan reflection argument, the pentagonal number theorem — is at bottom a cleverly chosen bijection. Algebra verifies; bijection explains.

Worked Example 1.1 (*The handshake reflex*). At a party of n people everyone shakes hands with everyone else exactly once. How many handshakes occur?

Solution. A handshake is an unordered pair of distinct people, so the set of handshakes is in bijection with the set of 2-element subsets of an n -element set, whose cardinality (Chapter 5) is $\binom{n}{2} = \frac{n(n-1)}{2}$. But we can also see this without the formula: line up the ordered pairs (a, b) with $a \neq b$; there are $n(n-1)$ of them, and the map $(a, b) \mapsto \{a, b\}$ is exactly two-to-one, so the number of unordered pairs is $n(n-1)/2$. This “count ordered, then divide by the redundancy” move — division as an inverse bijection — recurs throughout the subject. ■

1.2 Injections, surjections, and the two inequalities

Bijections give equalities; their weaker cousins give inequalities, and inequalities are often what a hard problem actually needs.

Proposition 1.3. *Let A, B be finite sets.*

- (a) *If there is an injection $A \rightarrow B$, then $|A| \leq |B|$.*
- (b) *If there is a surjection $A \rightarrow B$, then $|A| \geq |B|$.*
- (c) *If $|A| = |B|$ and $f: A \rightarrow B$ is either injective or surjective, then f is bijective.*

Proof. (a) An injection $f: A \rightarrow B$ is a bijection from A onto its image $f(A) \subseteq B$, and a subset of a finite set has cardinality at most that of the whole (an easy induction in the style of Theorem 1.2). (b) A surjection $f: A \rightarrow B$ admits a *section*: for each $b \in B$ choose one $a_b \in f^{-1}(b)$; then $b \mapsto a_b$ is an injection $B \rightarrow A$, and (a) applies. (c) Suppose f is injective but not surjective; then f is a bijection from A to a proper subset of B , giving $|A| < |B|$, a contradiction. The surjective case is symmetric, using the section from (b). □

Part (c) is the finite-set *pigeonhole equivalence*: for maps between equinumerous finite sets, injectivity, surjectivity and bijectivity coincide. This fails spectacularly for infinite sets ($n \mapsto n+1$ is an injection $\mathbb{N} \rightarrow \mathbb{N}$ that is not surjective), which is precisely Dedekind’s definition of infinitude. Chapter 15 will weaponize part (a) as the pigeonhole principle.

The Abstract Viewpoint • Counting is decategorification

A modern combinatorialist regards the passage from a set A to the number $|A|$ as a deliberate act of forgetting, called *decategorification*. The finite sets and bijections form a category; taking cardinality collapses this rich world onto the bare monoid $(\mathbb{N}, +, \times)$, where disjoint union becomes addition and Cartesian product becomes multiplication. A “bijective proof” of an identity $f(n) = g(n)$ is a lift of the numerical equation back up to the world of sets — a categorified identity — and it always carries strictly more information than the equation itself: it tells you not only *that* the two counts agree but *why*, functorially, naturally, and often preserving extra statistics. The research programme of “combinatorial species” (Joyal, 1981), which we will meet lightly in Chapter 13, takes this viewpoint as its foundation. When you learn to prefer a bijection to an induction, you are not being quaint; you are practising category theory with your bare hands.

1.3 The counting attitude: three questions

Before any formula, a trained counter interrogates a problem with three questions.

1.3.1 Does order matter?

“How many ways to award gold, silver, bronze among 10 athletes” and “how many ways to choose 3 medalists from 10” differ exactly in whether the selection is ordered. Ordered selections are *permutations* (Chapter 3); unordered selections are *combinations* (Chapter 5). Misreading this single bit is the most common exam error at the EAMCET/JEE Main level.

1.3.2 Is repetition allowed?

Choosing a 4-digit PIN allows the digit 7 to appear four times; choosing a committee does not allow a person to serve twice. The four answers to the pair (order?, repetition?) form the fundamental 2×2 selection table, which we preview now and prove across Chapters 3–6:

	Repetition allowed	No repetition
Ordered	n^r	$n^r = \frac{n!}{(n-r)!}$
Unordered	$\binom{n+r-1}{r}$	$\binom{n}{r}$

Every entry of this table is a theorem, and each will receive both an algebraic and a bijective proof.

1.3.3 Are the objects distinguishable?

Distributing 5 distinct books among 3 children is not the same problem as distributing 5 identical chocolates among 3 children. In Part II this question blossoms into Rota’s *twelvefold way* (Chapter 11), a $2 \times 2 \times 3$ master table of which the selection table above is one face.

Worked Example 1.2 (*EAMCET calibration*). From the digits 1, 2, 3, 4, 5, how many (i) 3-digit numbers, (ii) 3-digit numbers with distinct digits, (iii) 3-element subsets, can be formed?

Solution. (i) Ordered with repetition: $5^3 = 125$. (ii) Ordered without repetition: $5 \cdot 4 \cdot 3 = 60$. (iii) Unordered without repetition: $\binom{5}{3} = 10$. Three readings of one stem — the examiner is testing the three questions, not arithmetic. ■

1.4 Sets, multisets, sequences, and words

We fix vocabulary used relentlessly hereafter.

Definition 1.4. Let X be a set. A *sequence* (or *word*, or *string*) of length r over X is a function $[r] \rightarrow X$; order matters and repetition is allowed. An *r -permutation* of X is an injective such function. An *r -subset* is a subset of X of size r . An *r -multiset* on X is a function $m: X \rightarrow \mathbb{N}$ with $\sum_x m(x) = r$; the value $m(x)$ is the *multiplicity* of x .

The four notions are precisely the four cells of the selection table. Note the functional viewpoint: a word is a *function*, not a mystical arrangement. This viewpoint makes proofs mechanical. For instance, the number of words of length r over an n -set is the number of functions $[r] \rightarrow [n]$, and Chapter 2 computes it as n^r by the product rule.

Remark 1.5. Throughout, “ n distinct objects” means a set of size n ; “ n identical objects” means we count multisets or, equivalently, we count orbits under the symmetric group permuting the objects — a remark that will only become fully meaningful in Chapter 16, but which we plant now as a seed.

How examiners disguise the three questions

- *Words/numbers/PINs/flags* $\Rightarrow$ ordered. Watch for “distinct digits” toggling repetition off.
- *Committees/teams/hands of cards/subsets* $\Rightarrow$ unordered, no repetition.
- *Distributions of identical coins/balls* $\Rightarrow$ multisets (stars and bars, Chapter 6).
- *Seating around a table* $\Rightarrow$ ordered, then quotient by rotation (Chapter 4).
- In JEE Advanced multi-concept items, a single stem often chains two cells of the table; solve each stage separately and multiply (Chapter 2 justifies this).

Exercises

1. Exhibit an explicit bijection between the set of subsets of $[n]$ and the set of binary strings of length n . Conclude $|\mathcal{P}([n])| = 2^n$ using only the bijection principle and the product rule. [\[EAMCET/JEE\]](#)
2. Prove part (a) of Proposition 1.3 in full detail by induction on $|B|$. [\[ISI/CMI\]](#)
3. Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be injections between finite sets. Prove $|A| = |B|$ and that f is a bijection. (This is the finite Cantor–Schröder–Bernstein theorem; the infinite case is much subtler.) [\[CMI/OLYMPIAD\]](#)
4. A function $f: [n] \rightarrow [n]$ satisfies $f(f(x)) = x$ for all x . Show that the number of such f (involutions) satisfies $I(n) = I(n-1) + (n-1)I(n-2)$, by classifying what f does to the element n . [\[ISI/OLYMPIAD\]](#)
5. In how many ways can one choose an ordered pair (A, B) of subsets of $[n]$ with $A \subseteq B$? Answer with a bijection to ternary strings. [\[JEE ADVANCED\]](#)
6. (Erdős taste test.) Without any formula, argue that the number of ways to seat $2n$ people into n unlabeled pairs is $(2n-1)(2n-3)\cdots 3\cdot 1$, by asking whom person 1 is paired with, then whom the smallest-indexed unpaired person is paired with, and so on. [\[ISI/CSIR\]](#)

The Two Fundamental Principles

ELI5 • Menus and meals

A café offers 3 sandwiches and 4 juices. If lunch is *one sandwich or one juice*, you have $3 + 4 = 7$ lunches: the choices live in separate baskets, so you add. If lunch is *one sandwich and one juice*, you have $3 \times 4 = 12$ lunches: for each sandwich the whole shelf of juices reopens, so you multiply. “Or” means add (if the baskets don’t overlap); “and” means multiply (if the stages don’t interfere). Every counting formula in this book is built from these two moves, the way every word is built from letters.

2.1 The sum rule and finite additivity

Theorem 2.1 (Sum rule). *If $A_1, \dots, A_k$ are pairwise disjoint finite sets, then*

$$|A_1 \cup A_2 \cup \dots \cup A_k| = |A_1| + |A_2| + \dots + |A_k|.$$

Proof. It suffices to treat $k = 2$ and induct. Let $|A| = m$, $|B| = n$ with bijections $f: A \rightarrow [m]$, $g: B \rightarrow [n]$, and $A \cap B = \emptyset$. Define $h: A \cup B \rightarrow [m + n]$ by $h(x) = f(x)$ for $x \in A$ and $h(x) = m + g(x)$ for $x \in B$. Disjointness makes h well defined; injectivity and surjectivity are immediate since the ranges $[m]$ and $\{m + 1, \dots, m + n\}$ partition $[m + n]$. $\square$

The operational form of the sum rule is *casework*: to count a set, cut it into disjoint, exhaustive cases and add the case counts. The discipline lies in the two adjectives. *Disjoint*: no object may be counted in two cases (otherwise you have overcounted and must repair with inclusion–exclusion, Chapter 7). *Exhaustive*: every object must fall in some case.

Worked Example 2.1 (*Casework with a parity cut*). How many 3-digit numbers have strictly increasing digits?

Solution. A strictly increasing 3-digit string cannot contain 0 (a leading 0 is banned, and 0 can never follow a smaller digit). So we need strictly increasing words from $\{1, \dots, 9\}$, and each 3-subset of these nine digits arranges in exactly one increasing way — a bijection between the answers and 3-subsets. Hence $\binom{9}{3} = 84$. Notice the pattern: we did not enumerate cases at all; a bijection dissolved the problem. Casework is the tool of last resort, bijection the tool of first resort. $\blacksquare$

Worked Example 2.2 (*When casework is genuinely needed*). How many 4-digit numbers formed from digits $\{0, 1, 2, 3, 4, 5\}$ without repetition are divisible by 4? (JEE Main pattern)

Solution. Divisibility by 4 depends on the last two digits. List the two-digit tails $\overline{d_3d_4}$ with distinct digits from the set, divisible by 4: 04, 12, 20, 24, 32, 40, 52, 04... — carefully: 04, 12, 20, 24, 32, 40, 52 and also 04? Enumerate systematically: multiples of 4 between 00 and 55 with digits in $\{0, \dots, 5\}$, distinct: 04, 12, 20, 24, 32, 40, 52. Also 04 counted once; check 44 (repeat, reject), 00 (repeat, reject), 08, 16, 28, 36, 48 (digits outside set, reject). So 7 tails. Now count completions $\overline{d_1d_2}$: the two remaining slots draw from the 4 unused digits, with $d_1 \neq 0$. *Case A: tail contains 0* (tails 04, 20, 40: three tails). Remaining digits: 4 nonzero digits; d_1d_2 ordered

without repetition: $4 \cdot 3 = 12$ each, total $3 \cdot 12 = 36$. *Case B: tail avoids 0* (tails 12, 24, 32, 52: four tails). Remaining digits: 4 digits including 0; $d_1 \in 3$ nonzero choices, then $d_2 \in 3$ remaining: 9 each, total $4 \cdot 9 = 36$. Grand total $36 + 36 = 72$. ■

The example illustrates a second discipline: *choose the casework variable to be the constraint's home*. Divisibility by 4 lives in the last two digits, so we cased on tails, not on heads.

2.2 The product rule and the multiplication of stages

Theorem 2.2 (Product rule). $|A \times B| = |A| \cdot |B|$ for finite sets A, B . More generally, if a construction proceeds in k stages, where stage i can be completed in n_i ways regardless of the choices made at earlier stages (though the available options may differ, their number may not), then the number of completed constructions is $n_1 n_2 \cdots n_k$, provided distinct choice sequences yield distinct objects.

Proof. For the Cartesian product: $A \times B = \bigcup_{a \in A} \{a\} \times B$ is a disjoint union of $|A|$ blocks each bijective with B ; apply the sum rule. For the staged form: the choice sequences form a set bijective with a Cartesian product $[n_1] \times \cdots \times [n_k]$ (choose, at each stage, the index of the selected option in some fixed ordering of the currently available options — the hypothesis that the *number* n_i is stage-history-independent is exactly what makes this indexing well defined), and the final hypothesis makes the map from choice sequences to objects injective. □

Both italicized hypotheses are live wires.

The two ways the product rule fails

(F1) Stage counts depend on history. “Choose two adjacent squares on a chessboard, one at a time”: the number of neighbours of the first square is 2, 3, or 4 depending on where it is. Repair: refine into cases (corner, edge, interior) inside which the count is uniform.

(F2) Distinct choice sequences give the same object. Choosing a committee of 3 by picking members one at a time: the sequences (a, b, c) and (b, c, a) build the same committee. Repair: divide by the (constant!) number of sequences per object — the *shepherd's quotient* — which is legitimate precisely because the fibres of the sequence→object map all have the same size $3!$. This quotient principle is stated formally below.

Lemma 2.3 (Division/quotient principle). Let $\varphi: S \rightarrow T$ be a surjection between finite sets such that every fibre $\varphi^{-1}(t)$ has the same size d . Then $|T| = |S|/d$.

Proof. $S = \bigsqcup_{t \in T} \varphi^{-1}(t)$ is a disjoint union of $|T|$ blocks of size d ; the sum rule gives $|S| = d|T|$. □

Lemma 2.3 is humble here and heroic later: Lagrange's theorem in group theory, the orbit-stabilizer theorem (Chapter 16), and the formula $\binom{n}{r} = n^x/r!$ are all instances of it.

Worked Example 2.3 (Functions, injections, and the first two table entries). Let $|X| = r$, $|Y| = n$. Then the number of functions $X \rightarrow Y$ is n^r and the number of injections $X \rightarrow Y$ is $n^x = n(n-1) \cdots (n-r+1)$.

Solution. List $X = \{x_1, \dots, x_r\}$. A function is built in r stages: choose $f(x_1)$ (n ways), ..., choose $f(x_r)$ (n ways); choices never interfere, and distinct choice sequences are literally distinct functions. Product rule: n^r . For injections, stage i offers $n - i + 1$ options (all of Y minus the $i - 1$ used values); the *number* of options is history-independent even though the option *set* is not — exactly the subtlety Theorem 2.2 was phrased to permit. Product: $n(n-1) \cdots (n-r+1)$. ■

The Abstract Viewpoint • Sum and product as universal constructions

Disjoint union and Cartesian product are the coproduct and product in the category of sets: they are characterized not by their internal construction but by universal mapping properties (maps out of $A \sqcup B$ are pairs of maps out of A and out of B ; maps into $A \times B$ are pairs of maps in). Cardinality sends coproduct to $+$ and product to $\times$; in the language of Chapter 1, it is a homomorphism of “rig categories” onto the rig $(\mathbb{N}, +, \times)$. This is why the distributive law $a(b + c) = ab + ac$ has combinatorial content: it decategorifies the natural bijection $A \times (B \sqcup C) \cong (A \times B) \sqcup (A \times C)$. Generating functions (Chapter 12) exploit exactly this: they are cardinality homomorphisms enriched with a formal variable that remembers a statistic. When later we “multiply generating functions to concatenate structures,” we are re-running Theorem 2.2 in a graded setting.

2.3 Tree diagrams, and counting by recursion

For small stage-structures the product rule is a tree: the root branches into stage-1 options, each of which branches into stage-2 options, and the leaves are the constructed objects. The number of leaves is the product of branching factors — when those factors are uniform. When they are not uniform (failure F1), the tree still computes the answer: sum over leaves. Formally, this is counting by recursion:

Proposition 2.4 (Recursive counting). *Let $c(v)$ denote the number of leaves below a node v of a finite rooted tree. Then $c(\text{leaf}) = 1$ and $c(v) = \sum_{w \text{ child of } v} c(w)$; the answer to the counting problem is $c(\text{root})$.*

This trivial-looking proposition is the germ of dynamic programming and of every recurrence relation in Chapter 13. Competitive examinations at the ISI/CMI level frequently set problems whose only sane solution is to *define the right recursion*, not to find a closed form.

Worked Example 2.4 (*Recursion beats formulas*). How many binary strings of length 10 contain no two consecutive 1s? (ISI B.Stat pattern)

Solution. Let a_n count such strings of length n . A valid string ends in 0 (preceded by any valid string of length $n - 1$) or in 01 (preceded by any valid string of length $n - 2$); the cases are disjoint and exhaustive. Hence $a_n = a_{n-1} + a_{n-2}$, with $a_1 = 2$, $a_2 = 3$: a shifted Fibonacci sequence. Compute: 2, 3, 5, 8, 13, 21, 34, 55, 89, 144; so $a_{10} = 144$. Chapter 12 will solve this recurrence in closed form with generating functions, and Chapter 13 will explain why Fibonacci “must” appear: the transfer-matrix method. ■

2.4 Complementary counting

The sum rule read backwards is often the fastest tool in an examination hall: if $A \subseteq U$, then $|A| = |U| - |U \setminus A|$. Count the whole, subtract the bad.

Worked Example 2.5 (*JEE Main pattern*). How many 5-letter words from an alphabet of 26 letters (repetition allowed) contain at least one vowel?

Solution. All words: 26^5 . Vowel-free words: 21^5 . Answer: $26^5 - 21^5 = 11881376 - 4084101 = 7797275$. Attempting “at least one” directly — casing on exactly one, exactly two, ... — invites both slow arithmetic and inclusion–exclusion errors. The phrase *at least one* should trigger the complement reflex automatically. ■

Principle-selection heuristics under time pressure

- “at least one” $\rightarrow$ complement. “exactly k ” $\rightarrow$ choose the positions/objects for the special property, then fill the rest (product rule).
- Uniform stages $\rightarrow$ multiply. History-dependent stage *counts* $\rightarrow$ case first, multiply inside cases.
- Unordered target built by ordered process $\rightarrow$ divide by the fibre size (Lemma 2.3); verify the fibres really are equal-sized (they fail to be when objects can repeat — a trap resolved in Chapter 3, multiset permutations).
- Constraints that touch (adjacency, divisibility, “together”, “separated”) $\rightarrow$ transform first: glue blocks, fix tails, insert gaps (Chapters 3, 6), then apply the principles to the transformed problem.

Exercises

1. How many 6-digit numbers have digits strictly decreasing left to right? *[Hint: bijection with subsets of $\{0, 1, \dots, 9\}$]* [JEE MAIN]
2. Count the functions $f: [5] \rightarrow [5]$ with $f(1) \neq 1$. [EAMCET]
3. How many 4-digit numbers contain the digit 7 at least once? [EAMCET/JEE MAIN]
4. Count binary strings of length n with no two consecutive 1s that also begin and end with 0; express your answer via the sequence a_n of the worked example. [ISI]
5. A rook stands at the corner of an 8×8 board and moves only right or up, one square at a time. Show by recursion (no formulas) that the number of paths to the opposite corner is $\binom{14}{7}$, and only then explain the formula bijectively. [JEE ADVANCED]
6. Let D be the set of 10-digit strings using each digit 0–9 exactly once. How many members of D have the property that the digits 0 and 9 are not adjacent? *[Hint: complement + glue]* [JEE ADVANCED]
7. (Uniform-fibre vigilance.) Explain precisely why “choose 3 letters one at a time from the multiset $\{A, A, B, C\}$ and divide by $3!$ ” gives a wrong count of 3-letter selections, identifying which fibres of the ordering map have unequal sizes. [ISI/CMI]

Permutations: The Arithmetic of Arrangement

ELI5 • Photographs of a queue

Five friends queue for a photograph. The first spot can be filled by any of the 5; once someone stands there, the second spot has 4 candidates; then 3, 2, 1. So there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ photographs. The number $5 \times 4 \times 3 \times 2 \times 1$ is written $5!$ and read “five factorial.” Factorials count *arrangements*: all the ways to put things in a row when everything is different and every position matters.

3.1 Factorials and falling factorials

Definition 3.1. For $n \in \mathbb{N}$, define $0! = 1$ and $n! = n \cdot (n-1)!$. For $0 \leq r \leq n$, the *falling factorial* is $n^{\underline{r}} = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$, with $n^{\underline{0}} = 1$; the *rising factorial* is $n^{\overline{r}} = n(n+1) \cdots (n+r-1)$. The classical notation for $n^{\underline{r}}$ in Indian examinations is ${}^n P_r$ or $P(n, r)$.

Theorem 3.2. The number of r -permutations of an n -set (ordered selections of r distinct objects from n distinct objects) is $n^{\underline{r}}$. In particular the number of full arrangements (bijections of the set with itself, i.e. permutations proper) is $n!$.

Proof. An r -permutation is an injection $[r] \rightarrow [n]$; the product-rule computation of Chapter 2 gives $n(n-1) \cdots (n-r+1)$. Taking $r = n$, Proposition 1.2(c) upgrades injections to bijections. $\square$

Convention 3.3 (Why $0! = 1$). Three independent reasons force the convention. (i) The recursion $n! = n(n-1)!$ demands $1! = 1 \cdot 0!$. (ii) There is exactly one bijection $\emptyset \rightarrow \emptyset$ (the empty function), so the count of arrangements of nothing is 1, not 0. (iii) Every formula of Chapter 5 — e.g. $\binom{n}{0} = n!/(0!n!) = 1$ — would break otherwise. Empty products equal 1 for the same reason empty sums equal 0: they are identity elements.

3.1.1 Growth and Stirling’s formula

Factorials outgrow every exponential: $n! > (n/e)^n$ for all $n \geq 1$, and more precisely

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad (n \rightarrow \infty),$$

Stirling’s asymptotic. A cheap version sufficient for CSIR/olympiad estimates follows from comparing $\ln n! = \sum_{k=1}^n \ln k$ with $\int_1^n \ln x \, dx = n \ln n - n + 1$:

Proposition 3.4. $e \left(\frac{n}{e}\right)^n \leq n! \leq en \left(\frac{n}{e}\right)^n$ for all $n \geq 1$.

Proof. Since $\ln$ is increasing, $\int_{k-1}^k \ln x \, dx \leq \ln k \leq \int_k^{k+1} \ln x \, dx$. Summing $k = 2, \dots, n$ on the left: $\ln n! \geq \int_1^n \ln x \, dx = n \ln n - n + 1$. Summing $k = 1, \dots, n$ on the right and adjusting:

$\ln n! \leq \int_1^{n+1} \ln x \, dx \leq (n+1) \ln(n+1) - n$; a slightly more careful pairing (sum $k = 2, \dots, n$ against $\int_{k-1}^k$ shifted) yields $\ln n! \leq n \ln n - n + 1 + \ln n$. Exponentiate. $\square$

Examinations rarely ask Stirling directly, but ISI, CMI and CSIR routinely ask for the *comparisons* it enables: which of $100!$, 50^{100} , 2^{500} is largest, or the asymptotics of $\binom{2n}{n}$ (Chapter 9 shows $\binom{2n}{n} \sim 4^n / \sqrt{\pi n}$).

3.2 Arrangements with constraints: the four transforms

Constrained arrangements are solved by transforming to unconstrained ones. Four transforms cover the examination universe.

3.2.1 The glue (block) transform: “together”

To arrange n objects with a specified subset S of size k required to be consecutive: fuse S into a single super-object, arrange the resulting $n - k + 1$ objects in $(n - k + 1)!$ ways, then arrange S internally in $k!$ ways. Total $(n - k + 1)! k!$. The two stages are independent, so the product rule applies cleanly.

3.2.2 The gap (insertion) transform: “no two together”

To arrange m objects of type A and k of type B, all distinct, with no two B’s adjacent: first arrange the A’s ($m!$ ways), creating $m + 1$ gaps (including both ends); choose k gaps and populate them with B’s, one per gap: $(m + 1)^k$ ways. Total $m! (m + 1)^k$, valid iff $k \leq m + 1$.

Worked Example 3.1 (*JEE Main staple*). In how many ways can 5 boys and 3 girls stand in a row so that no two girls are adjacent?

Solution. Boys: $5! = 120$; gaps: 6; place girls: $6^3 = 216$. Answer $120 \times 216 = 25920$. $\blacksquare$

3.2.3 The complement transform: “not together”

For exactly two objects forbidden to be adjacent, complement is faster than gaps: (all arrangements) $-$ (arrangements with the pair glued) $= n! - 2(n - 1)!$.

3.2.4 The position-fixing transform: boundary constraints

Constraints pinned to positions (first digit nonzero; word begins and ends with a vowel; alternating genders) are handled by filling the constrained positions first, then the free positions, and multiplying — always ordering the stages from most constrained to least, so that stage counts remain uniform.

Worked Example 3.2 (*Alternation, EAMCET pattern*). In how many ways can 4 men and 4 women be seated in a row alternately?

Solution. Pattern MWMWMWMW or WMWMWMWM: 2 patterns; each fills men’s slots in $4!$ ways and women’s in $4!$: $2 \cdot 24 \cdot 24 = 1152$. $\blacksquare$

3.3 Permutations of multisets

When objects repeat, naive division must be replaced by the multinomial count.

Theorem 3.5 (Multiset permutations). *The number of distinct arrangements of a multiset with n_i copies of type i , $i = 1, \dots, k$, $n = n_1 + \dots + n_k$, is the multinomial coefficient*

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! n_2! \dots n_k!}.$$

Proof 1 (quotient). Temporarily label the copies to make all n objects distinct: $n!$ arrangements. The forgetting map (erase labels) sends an arrangement of labelled objects to an arrangement of the multiset; each multiset arrangement is hit by exactly $n_1! n_2! \dots n_k!$ labelled ones (independently permute the labels within each type), a uniform fibre size. Lemma 2.3 divides. $\square$

Proof 2 (staged choice). Choose which n_1 of the n positions carry type 1: $\binom{n}{n_1}$ ways; then which n_2 of the remaining $n - n_1$ carry type 2: $\binom{n - n_1}{n_2}$; and so on. The telescoping product $\binom{n}{n_1} \binom{n - n_1}{n_2} \dots \binom{n_k}{n_k}$ collapses to $n! / (n_1! \dots n_k!)$ upon writing each factor with factorials. $\square$

Proof 2 quietly proves more: it identifies multiset arrangements with *ordered set partitions of the position set with prescribed block sizes*, a bijection we reuse in Chapters 6 and 11.

Worked Example 3.3 (*The word standard*). Count the arrangements of the letters of MISSISSIPPI; then count those in which no two I's are adjacent.

Solution. Letters: M $\times$ 1, I $\times$ 4, S $\times$ 4, P $\times$ 2, $n = 11$. Total: $\frac{11!}{1! 4! 4! 2!} = 34650$. For the constraint, arrange the 7 non-I letters: $\frac{7!}{4! 2!} = 105$; the 8 gaps receive the four identical I's in $\binom{8}{4} = 70$ ways (identical objects into distinct gaps, at most one per gap, is a plain combination). Answer $105 \times 70 = 7350$. Note the seamless splice of the gap transform with multiset counting. $\blacksquare$

Worked Example 3.4 (*Rank of a word, JEE Advanced pattern*). If all arrangements of the letters of RANDOM are listed alphabetically, find the rank of the word RANDOM.

Solution. Alphabetical order of letters: A, D, M, N, O, R. Count words strictly earlier. First letter earlier than R: any of A, D, M, N, O (5 choices) followed by any arrangement of the remaining 5 letters: $5 \cdot 5! = 600$. First letter R; second earlier than A: none (A is minimal): 0. Prefix RA; third earlier than N: from remaining $\{D, M, N, O\}$, letters below N: D, M (2), times 3!: 12. Prefix RAN; fourth earlier than D: remaining $\{D, M, O\}$; below D: none: 0. Prefix RAND; fifth earlier than O: remaining $\{M, O\}$; below O: M (1), times 1!: 1. Prefix RANDO; sixth earlier than M: remaining $\{M\}$: 0. Words before: $600 + 12 + 1 = 613$; rank = 614. The algorithm — “for each position, count smaller available letters times factorial of the tail” — is the *factorial number system* in disguise and deserves to be memorized as an algorithm, not an anecdote. $\blacksquare$

The Abstract Viewpoint • Permutations as a group; multisets as orbits

The set S_n of bijections $[n] \rightarrow [n]$ is a group under composition, the *symmetric group*, of order $n!$; it is the mother object of Part II. Theorem 3.5 is our first orbit count: the group $S_{n_1} \times \dots \times S_{n_k}$ acts on the $n!$ labelled arrangements by permuting identical items, all orbits have full size $n_1! \dots n_k!$ (the action is *free*), and arrangements-of-a-multiset are precisely the orbits. Burnside's lemma (Chapter 16) generalizes this to actions that are not free — which is exactly what goes wrong in circular and bracelet problems, where naive division fails. Meanwhile the multinomial coefficient itself is the dimension $\dim_{\mathbb{C}}$ of the weight space of weight $(n_1, \dots, n_k)$ in the n -th tensor power of $\mathbb{C}^k$ — the first hint that counting formulas are

shadows of representation theory, a theme CSIR aspirants will meet again in the Schur–Weyl story.

3.4 Permutations as bijections: cycle structure

For Part II we need the structural, not merely enumerative, view.

Definition 3.6. Let $\sigma \in S_n$. The *cycle* $(a_1 a_2 \cdots a_\ell)$ denotes the permutation mapping $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_\ell \mapsto a_1$ and fixing everything else. Every σ factors uniquely (up to order) into disjoint cycles: follow each element until it returns. The multiset of cycle lengths is the *cycle type* of σ .

Theorem 3.7 (Cycle-type census). The number of permutations in S_n with c_j cycles of length j ($\sum_j j c_j = n$) is

$$\frac{n!}{\prod_{j \geq 1} j^{c_j} c_j!}.$$

Proof. Write a template with blank cycles of the prescribed lengths and fill the n blanks with $1, \dots, n$: $n!$ fillings. Each permutation is produced by many fillings: each j -cycle can be written starting at any of its j elements (j^{c_j} redundancy over all j -cycles) and cycles of equal length can be listed in any order ($c_j!$ redundancy). Fibres are uniform of size $\prod_j j^{c_j} c_j!$; divide. $\square$

Worked Example 3.5. How many permutations of $[6]$ consist of two disjoint 3-cycles?

Solution. $c_3 = 2$: $\frac{6!}{3^2 \cdot 2!} = \frac{720}{18} = 40$. ■

Cycle structure powers Chapter 16 (Burnside sums are sums over cycle types), Chapter 12 (the exponential formula), and standard CSIR questions on the order of a permutation: the order of σ is lcm of its cycle lengths, whence the maximal order in S_n (Landau’s function) grows like $e^{(1+o(1))\sqrt{n \ln n}}$.

Constraint-transform decision chart

Given an arrangement problem, ask in order: (1) Repetition among objects? $\rightarrow$ multinomial denominator. (2) “Together”? $\rightarrow$ glue; “no two together”? $\rightarrow$ gaps; “these two apart”? $\rightarrow$ complement. (3) Position-pinned constraints? $\rightarrow$ fill pinned slots first. (4) Circular table? $\rightarrow$ Chapter 4 before anything else. (5) Rank/dictionary problems? $\rightarrow$ factorial number system algorithm. Combine transforms in the order (4)(2)(3)(1): geometry first, adjacency second, pins third, identical-object division last.

Exercises

1. How many arrangements of BANANA exist? How many have the three A’s mutually non-adjacent? [EAMCET]
2. How many 9-letter arrangements of ALLAHABAD begin and end with A? [JEE MAIN]
3. Find the number of permutations of $[7]$ having exactly two cycles. [Hint: Theorem 3.7; sum over cycle types $(6, 1), (5, 2), (4, 3)$; you have discovered the Stirling number $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$] [ISI/CSIR]

4. Prove that the number of permutations of $[n]$ that are involutions (order dividing 2) equals $\sum_k \binom{n}{2k} \frac{(2k)!}{2^k k!}$, and reconcile with Exercise 1.4. [\[CMI/CSIR\]](#)
5. Find the rank of MOTHER among all arrangements of its letters in dictionary order. [\[JEE ADVANCED\]](#)
6. Show that the number of ways to seat n couples in a row of $2n$ chairs so that every couple sits together is $2^n n!$, and so that *no* couple sits together requires inclusion–exclusion (set it up; solve after Chapter 7). [\[JEE ADVANCED/ISI\]](#)
7. Let $f(n)$ be the number of permutations of $[n]$ with all cycle lengths ≤ 2 . Re-derive $f(n) = f(n-1) + (n-1)f(n-2)$ from cycle structure. [\[ISI\]](#)
8. (Olympiad.) Prove that a permutation and its inverse have the same cycle type, and that two permutations are conjugate in S_n iff they share a cycle type. [\[OLYMPIAD/NBHM\]](#)

Circular Arrangements, Necklaces, and the First Taste of Symmetry

ELI5 • Round tables have no head

Seat three friends A, B, C in a row: ABC, ACB, BAC, BCA, CAB, CBA — six ways. Now seat them around a round table. Spin the table: ABC becomes BCA becomes CAB. Those three “different” rows are the *same* circle, just viewed after a rotation. Six rows collapse into pairs of... actually triples: $6 \div 3 = 2$ genuinely different circles (clockwise ABC and clockwise ACB). Round tables erase the notion of a first seat, so we divide by the number of spins.

4.1 The basic circular count

Theorem 4.1. *The number of ways to seat n distinct people around a circular table, where arrangements differing by a rotation are identified, is $(n - 1)!$.*

Proof 1 (anchor). Rotational equivalence is destroyed by fixing one person — say person 1 — in a distinguished seat; every circular arrangement has exactly one representative with person 1 in that seat. The remaining $n - 1$ people fill the remaining seats linearly: $(n - 1)!$. $\square$

Proof 2 (quotient). There are $n!$ seatings of labelled people into n labelled seats. The rotation group C_n (of order n) acts on seatings; two seatings are equivalent iff related by a rotation. Because the people are *distinct*, no nontrivial rotation fixes any seating (a rotation by $k \neq 0$ moves person 1 to a different seat), so the action is free: all orbits have size exactly n . Lemma 2.3: $n!/n = (n - 1)!$. $\square$

Proof 2 is deliberately heavier, because its hypothesis — *freeness* — is exactly what fails in the beads-and-necklaces problems below, where naive division gives wrong (even non-integral) answers. Circular counting is the student’s first encounter with the difference between dividing by a group order (valid only for free actions) and genuinely averaging over a group (Burnside, Chapter 16).

Worked Example 4.1 (*EAMCET/JEE Main pattern*). In how many ways can 5 men and 5 women sit around a round table so that men and women alternate?

Solution. Seat the men circularly: $(5 - 1)! = 24$. The men now create 5 genuinely distinct gaps (distinct because the men are distinct — rotational symmetry is already spent), into which the 5 women go linearly: $5! = 120$. Answer $24 \times 120 = 2880$. Anchoring must be done *once*: a common error seats women also circularly and divides again. $\blacksquare$

Worked Example 4.2 (*Garlands and bracelets*). How many distinct garlands can be made from 8 distinct flowers, if a garland can be rotated *and flipped over*?

Solution. Rotations alone: $(8 - 1)! = 5040$. Flipping identifies each garland with its mirror image; because the flowers are distinct, no garland is its own mirror image under any reflection composed

with rotation... but this claim needs care, and here it is safe: an arrangement of ≥ 3 distinct beads is never fixed by a reflection (a reflection either fixes a bead and swaps its neighbours, forcing two distinct flowers to be equal, or swaps beads in pairs similarly). Hence flipping pairs up the 5040 arrangements freely: $5040/2 = 2520$. ■

Row vs. circle vs. free circle

Distinct objects, all n used: row $n!$; circle (rotations identified) $(n-1)!$; free circle / garland (rotations and reflections identified) $\frac{1}{2}(n-1)!$ for $n \geq 3$. For *selections*: to seat r of n people in a circle of r chairs: $\binom{n}{r}(r-1)! = n^r/r$.

4.2 When division fails: identical beads

Consider necklaces of n beads drawn from k colours, rotations identified. The labelled count is k^n ; the tempting answer k^n/n is generally not even an integer ($k = n = 2$: $4/2 = 2$, but the necklaces are BB, BW, WW: three of them). The action of C_n on colourings is not free: the monochromatic colouring is fixed by *every* rotation. The correct tool averages fixed points:

Theorem 4.2 (Necklace-counting preview; proved in Chapter 16). *The number of n -bead necklaces with k available colours, under rotation, is*

$$\frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d},$$

where φ is Euler's totient.

For $n = 2, k = 2$: $\frac{1}{2}(\varphi(1)2^2 + \varphi(2)2^1) = \frac{1}{2}(4 + 2) = 3$. Correct. We flag the formula now for three reasons: JEE Advanced has flirted with small cases; ISI/CMI interviews love the failure of naive division; and the formula motivates the entire architecture of Chapter 16.

The Abstract Viewpoint • Quotients are not always sets of orbits of equal size

The pattern behind this chapter: a counting problem with symmetry is a set X with a group action $G \curvearrowright X$, and “essentially different configurations” means orbits X/G . When the action is free, $|X/G| = |X|/|G|$ — this is Lagrange/quotient counting and covers Theorem 4.1. When it is not, the honest formula is Burnside–Cauchy–Frobenius: $|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$, an average of fixed points. Deeper still, the orbit–stabilizer theorem $|\text{Orb}(x)| \cdot |\text{Stab}(x)| = |G|$ explains *why* naive division fails: orbits of highly symmetric configurations are small, precisely by the size of their stabilizer. Chapter 16 builds this properly and Pólya's theorem upgrades it to count with colours weighted by variables — at which point necklaces, bracelets, chemical isomers and switching functions all fall to a single machine.

4.3 Circular problems with structure

Worked Example 4.3 (*Blocks on a circle*). In how many ways can 4 couples sit around a round table so that each couple sits together?

Solution. Glue couples: 4 blocks in a circle: $(4-1)! = 6$; internal orders $2^4 = 16$; answer 96. ■

Worked Example 4.4 (*Gaps on a circle, JEE Advanced pattern*). In how many ways can 6 boys and 3 girls sit around a round table so that no two girls are adjacent?

Solution. Boys circularly: $5! = 120$; the boys bound 6 gaps; choose and fill 3 gaps with distinct girls: $6^3 = 216$; answer 14400. ■

Worked Example 4.5 (*Numbered seats undo the circle*). Ten people sit at a round table whose chairs are numbered 1 to 10. How many seatings?

Solution. $10!$. Numbered chairs are distinguishable positions; rotation is no longer a symmetry of the *problem*. The circle in the picture is decoration; the mathematics sees a row. Always ask: what data does a rotation actually preserve? ■

Circular checklist

(1) Are seats labelled? If yes, it is a linear problem: $n!$ -type counts. (2) Rotations only, or also reflections? “Necklace/garland which can be turned over” $\Rightarrow$ divide the distinct-bead answer by 2 (valid for distinct beads, $n \geq 3$). (3) Distinct occupants $\Rightarrow$ anchor one occupant, count the rest linearly. Identical items anywhere $\Rightarrow$ suspect non-free action; use gap methods around anchored distinct items, or (beyond JEE) Burnside. (4) Alternation/adjacency on circles: anchor first, then gaps; never divide twice.

Exercises

1. In how many ways can 7 people sit around a round table if two particular people must not sit together? [EAMCET]
2. Twenty delegates are to be seated at a round table. Find the number of seatings in which three particular delegates are mutually separated (no two of the three adjacent). [JEE ADVANCED]
3. How many necklaces of 6 beads can be strung using beads of 2 colours, rotations only? Verify Theorem 4.2 by exhaustive listing. [ISI/CMI]
4. Prove carefully the claim used for garlands: for $n \geq 3$ distinct beads, no circular arrangement equals its own reflection. Exhibit the failure at $n = 2$. [CMI]
5. $2n$ people sit around a round table; count the seatings in which a specified pair sits diametrically opposite. [JEE MAIN]
6. (Ménage teaser.) Set up — do not yet solve — the count of circular seatings of n couples, alternating gender, with no couple adjacent. Chapter 7 finishes it. [OLYMPIAD]

Combinations and the Identity Calculus of $\binom{n}{r}$

ELI5 • Handfuls, not queues

From a jar of 5 differently coloured marbles you grab 2 in one handful. A handful has no first or second marble — it is just a pair. Each pair could have been picked up in 2 orders, so the $5 \times 4 = 20$ ordered grabs collapse into $20 \div 2 = 10$ handfuls. The symbol $\binom{5}{2}$, read “5 choose 2,” counts handfuls: selections where only *membership* matters.

5.1 Definition and the two formulas

Definition 5.1. For integers $0 \leq r \leq n$, $\binom{n}{r}$ denotes the number of r -element subsets of an n -element set. Set $\binom{n}{r} = 0$ for $r < 0$ or $r > n$. The examination notation nC_r means the same object.

Theorem 5.2. $\binom{n}{r} = \frac{n^{\underline{r}}}{r!} = \frac{n!}{r!(n-r)!}$.

Proof. The map (ordered r -selection) $\mapsto$ (its underlying set) from r -permutations onto r -subsets has uniform fibres of size $r!$ (each subset orders in $r!$ ways, by Theorem 3.2 applied to the subset). Lemma 2.3. $\square$

An immediate but psychologically important corollary: the product of any r consecutive positive integers is divisible by $r!$, since $n^{\underline{r}}/r! = \binom{n}{r} \in \mathbb{Z}$. Divisibility facts proved by “it counts something” pervade Chapter 8.

5.2 The identity calculus

The binomial coefficients satisfy an interlocking web of identities. We prove the load-bearing ones, each with a combinatorial (bijective/double-counting) proof — the proof style ISI, CMI and olympiads reward — and note the algebraic shadow.

Theorem 5.3 (Symmetry). $\binom{n}{r} = \binom{n}{n-r}$.

Proof. Complementation $S \mapsto [n] \setminus S$ is an involutive bijection between r -subsets and $(n-r)$ -subsets. $\square$

Theorem 5.4 (Pascal’s rule). $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Proof. Split the r -subsets of $[n]$ by whether they contain the element n : those that do correspond to $(r-1)$ -subsets of $[n-1]$; those that don’t are r -subsets of $[n-1]$. Sum rule. $\square$

Pascal’s rule generates the triangle and, iterated, everything else; but individual identities have better individual proofs.

Theorem 5.5 (Committee-chair identity). $r \binom{n}{r} = n \binom{n-1}{r-1}$; equivalently $\binom{n}{r} = \frac{n}{r} \binom{n-1}{r-1}$.

Proof. Count pairs (committee of r , chair within it) two ways: choose committee then chair: $\binom{n}{r} \cdot r$; choose chair from the population then the rest of the committee: $n \cdot \binom{n-1}{r-1}$. $\square$

Theorem 5.6 (Sum of a row; subsets by parity). $\sum_{r=0}^n \binom{n}{r} = 2^n$, and for $n \geq 1$, $\sum_{r \text{ even}} \binom{n}{r} = \sum_{r \text{ odd}} \binom{n}{r} = 2^{n-1}$.

Proof. First: both sides count all subsets of $[n]$ (right side by the indicator bijection of Exercise 1.1). Second: the map $S \mapsto S \triangle \{1\}$ (toggle element 1) is a bijection between even-sized and odd-sized subsets. $\square$

Theorem 5.7 (Vandermonde's identity). $\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{k} \binom{n}{r-k}$.

Proof. A committee of r from m men and n women contains some number k of men; case on k and apply sum and product rules. The special case $m = n = r$ with symmetry gives $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$. $\square$

Theorem 5.8 (Hockey stick / upper summation). $\sum_{m=r}^n \binom{m}{r} = \binom{n+1}{r+1}$.

Proof. Count $(r+1)$ -subsets of $[n+1]$ by their largest element $m+1$: the rest of the subset is an r -subset of $[m]$. $\square$

Theorem 5.9 (Absorption of alternating signs). For $n \geq 1$: $\sum_{r=0}^n (-1)^r \binom{n}{r} = 0$, and partially,

$$\sum_{r=0}^k (-1)^r \binom{n}{r} = (-1)^k \binom{n-1}{k}.$$

Proof. The full sum is Theorem 5.6's parity statement. The partial sum follows by induction on k using Pascal: the sum telescopes since $(-1)^r \binom{n}{r} = (-1)^r \binom{n-1}{r} - (-1)^{r-1} \binom{n-1}{r-1}$. $\square$

Worked Example 5.1 (Identity recognition, JEE Advanced pattern). Evaluate $\sum_{k=1}^n k \binom{n}{k}^2$.

Solution. By committee-chair, $k \binom{n}{k} = n \binom{n-1}{k-1}$, so the sum is $n \sum_k \binom{n-1}{k-1} \binom{n}{k} = n \sum_k \binom{n-1}{k-1} \binom{n}{n-k}$, and Vandermonde with upper indices $(n-1) + n$ choosing $n-1$ gives $n \binom{2n-1}{n-1}$. $\blacksquare$

Worked Example 5.2 (Double counting as a proof style, ISI pattern). Show that $\binom{n}{2} \binom{n-2}{2} = 3 \binom{n}{4}$. — more precisely, determine the constant c with $\binom{n}{2} \binom{n-2}{2} = c \binom{n}{4}$.

Solution. The left side counts ordered pairs (first a 2-subset, then a 2-subset of the rest), i.e. ways to split off two disjoint labelled pairs from $[n]$. Each 4-subset underlies exactly $\binom{4}{2} = 6$ such ordered pairs-of-pairs (choose which two of its elements form the first pair, but the order of the two pairs matters and is already encoded: $\binom{4}{2} = 6$). Hence $c = 6$: $\binom{n}{2} \binom{n-2}{2} = 6 \binom{n}{4}$. $\blacksquare$

5.3 The binomial theorem

Theorem 5.10 (Binomial theorem). *For any commutative ring elements x, y and $n \in \mathbb{N}$,*

$$(x + y)^n = \sum_{r=0}^n \binom{n}{r} x^r y^{n-r}.$$

Proof. Expand the product of n factors $(x + y)(x + y) \cdots (x + y)$ by distributivity: each monomial arises by choosing, from each factor, either x or y . The monomial $x^r y^{n-r}$ arises once for each choice of which r factors donate x — that is, once per r -subset of the factor set: $\binom{n}{r}$ times. $\square$

This proof is the distributive law read combinatorially — the Chapter 2 abstract-viewpoint box made concrete. Every “binomial identity by substituting $x = \pm 1$ ” trick is a specialization: $x = y = 1$ recovers Theorem 5.6; $x = -1, y = 1$ the alternating sum; differentiating in x and setting $x = y = 1$ gives $\sum r \binom{n}{r} = n2^{n-1}$, the calculus shadow of committee-chair. The multinomial theorem generalizes with Theorem 3.3’s coefficients:

$$(x_1 + \cdots + x_k)^n = \sum_{n_1 + \cdots + n_k = n} \binom{n}{n_1, \dots, n_k} x_1^{n_1} \cdots x_k^{n_k},$$

by the identical argument with k -way choices per factor.

The Abstract Viewpoint • Three faces of $\binom{n}{r}$

(1) *Enumerative*: cardinality of the subset lattice’s rank- r level. (2) *Algebraic*: coefficient extraction, $\binom{n}{r} = [x^r](1+x)^n$ — the germ of generating-functionology, where identities become algebra of power series (Vandermonde is literally $(1+x)^m(1+x)^n = (1+x)^{m+n}$ compared coefficientwise). (3) *Polynomial/functorial*: for fixed r , $\binom{x}{r} := x^{\underline{r}}/r!$ is a polynomial in x of degree r , and $\{\binom{x}{r}\}_{r \geq 0}$ is a $\mathbb{Z}$ -basis of the ring of integer-valued polynomials — the correct basis for the finite difference calculus, where $\Delta \binom{x}{r} = \binom{x}{r-1}$ mirrors $\frac{d}{dx} \frac{x^r}{r!} = \frac{x^{r-1}}{(r-1)!}$. Newton’s forward-difference formula $f(x) = \sum_r \Delta^r f(0) \binom{x}{r}$ is Taylor’s theorem for this calculus, and it explains why polynomial-sum problems (ISI loves $\sum_{k=1}^n k^3$ -type identities) are best solved in the binomial basis. q -deform everything and you obtain Gaussian binomials (Chapter 19), which count subspaces of $\mathbb{F}_q^n$ the way $\binom{n}{r}$ counts subsets.

5.4 Estimates and unimodality

Proposition 5.11. *For fixed n , the sequence $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ strictly increases up to the middle and strictly decreases after (with a plateau of two equal maxima iff n is odd). Moreover*

$$\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n, \quad \text{and} \quad \left(\frac{n}{r}\right)^r \leq \binom{n}{r} \leq \left(\frac{en}{r}\right)^r.$$

Proof. Ratio test: $\binom{n}{r+1}/\binom{n}{r} = (n-r)/(r+1) > 1 \iff r < \frac{n-1}{2}$. For the central estimate: $\binom{2n}{n}$ is the largest of the $2n+1$ entries in row $2n$, whose total is 4^n ; hence it is at least the average $4^n/(2n+1)$ and at most the total. For the last: lower bound from $\binom{n}{r} = \prod_{i=0}^{r-1} \frac{n-i}{r-i} \geq \prod (n/r)$ term by term (each $\frac{n-i}{r-i} \geq \frac{n}{r}$); upper bound from $\binom{n}{r} \leq \frac{n^r}{r!}$ and $r! \geq (r/e)^r$ (Proposition 3.4). $\square$

These inequalities are the daily bread of CSIR/NBHM asymptotic questions and olympiad-style estimates (e.g. Erdős’s proof of Bertrand’s postulate runs entirely on $\binom{2n}{n}$ bounds).

Manipulation drill table

Reflexes worth automatizing: $\binom{n}{r}/\binom{n}{r-1} = \frac{n-r+1}{r}$ (ratio questions); $\sum r\binom{n}{r} = n2^{n-1}$, $\sum r^2\binom{n}{r} = n(n+1)2^{n-2}$ (differentiate twice); $\sum \frac{1}{r+1}\binom{n}{r} = \frac{2^{n+1}-1}{n+1}$ (integrate); $\binom{n}{r}$ max at $r = \lfloor n/2 \rfloor$; greatest binomial coefficient in $(1+x)^n$ for given $x > 0$ at $r = \lfloor (n+1)x/(1+x) \rfloor$ (JEE Advanced “numerically greatest term”).

Exercises

1. Prove $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ by counting committee–subcommittee pairs. [JEE MAIN]
2. Evaluate $\sum_{k=0}^n \binom{n}{k}\binom{n}{k+2}$ via Vandermonde. [JEE ADVANCED]
3. Show combinatorially: $\binom{2n}{2} = 2\binom{n}{2} + n^2$. [ISI]
4. Prove the hockey stick identity a second time, by telescoping with Pascal’s rule. [JEE MAIN]
5. Prove $\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n}{n-k} = (-1)^{n/2} \binom{n}{n/2}$ for even n and 0 for odd n , by extracting the coefficient of x^n in $(1-x^2)^n$. [ISI/CMI]
6. For which r is $\binom{50}{r}$ greatest? For which term is the expansion of $(2+3x)^{20}$ at $x=1$ numerically greatest? [EAMCET/JEE]
7. Prove that $\binom{2n}{n}$ is never a perfect square for $n \geq 1$ is *hard*; instead prove the accessible fact that $\binom{2n}{n}$ is even for $n \geq 1$, bijectively. [Hint: pair each n -subset with its complement] [CMI/OLYMPIAD]
8. (NBHM style.) Show that $\sum_{r=0}^n \binom{n}{r}^2 x^r$ has all real roots for $x \dots$ is false as stated; instead prove $\sum_r \binom{n}{r}^2 = \binom{2n}{n}$ and deduce $\binom{2n}{n} < 4^n/\sqrt{2n}$ for $n \geq 1$ from Cauchy–Schwarz applied to the row of squares. [NBHM/CSIR]

Distributions: Stars, Bars, and Compositions

ELI5 • Ten chocolates, three children

You must hand out 10 identical chocolates to 3 children. Lay the chocolates in a row: ★★★★★★★★. Now drop in 2 dividers: everything left of the first divider goes to Anu, between the dividers to Balu, right of the second to Charu. For instance ★★★ | ★★★★★ | ★★ means (3, 5, 2). Every handout is a picture of 12 symbols — 10 stars and 2 bars — and every such picture is a handout. So the number of handouts equals the number of ways to choose which 2 of the 12 positions hold bars: $\binom{12}{2} = 66$.

6.1 The stars-and-bars theorems

Theorem 6.1 (Nonnegative solutions). *The number of solutions in nonnegative integers of $x_1 + x_2 + \cdots + x_k = n$ is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$. Equivalently, this counts n -multisets on a k -set — the fourth cell of the selection table.*

Proof. Encode a solution as a word of n stars and $k-1$ bars: x_1 stars, bar, x_2 stars, bar, $\dots$, x_k stars. The encoding is a bijection onto all arrangements of the multiset $\{\star^n, |^{k-1}\}$, of which there are $\binom{n+k-1}{k-1}$ by Theorem 3.5 (or directly: choose bar positions). For the multiset statement: a multiset with multiplicities $m(1), \dots, m(k)$ is a solution $x_i = m(i)$. $\square$

Theorem 6.2 (Positive solutions; compositions). *The number of solutions in positive integers of $x_1 + \cdots + x_k = n$ is $\binom{n-1}{k-1}$. Consequently the total number of compositions of n (ordered sums of positive parts, any number of parts) is 2^{n-1} .*

Proof. Substitute $y_i = x_i - 1 \geq 0$: the equation becomes $y_1 + \cdots + y_k = n - k$, counted by Theorem 6.1 as $\binom{n-1}{k-1}$. Alternatively and better: write n as a row of n stars with $n-1$ internal gaps; a composition into k parts is a choice of $k-1$ gaps to cut. Summing $\sum_{k=1}^n \binom{n-1}{k-1} = 2^{n-1}$ — or note directly that a composition is an arbitrary subset of the cut positions. $\square$

The substitution trick generalizes to any lower bounds:

Corollary 6.3 (Shifted bounds). *The number of integer solutions of $x_1 + \cdots + x_k = n$ with $x_i \geq a_i$ is $\binom{n - \sum a_i + k - 1}{k-1}$ (and 0 if $n < \sum a_i$).*

Upper bounds ($x_i \leq b_i$) are *not* handled by substitution; they require inclusion–exclusion, and we postpone them to Chapter 7, where they constitute one of its principal applications.

Worked Example 6.1 (JEE Main pattern). Find the number of nonnegative integer solutions of $x + y + z + w = 20$ with $x \geq 1$, $y \geq 2$, $z \geq 3$.

Solution. Shift: $n' = 20 - (1 + 2 + 3 + 0) = 14$, $k = 4$: $\binom{14+3}{3} = \binom{17}{3} = 680$. ■

Worked Example 6.2 (*Monomials and dice*). (i) How many monomials $x^a y^b z^c$ of total degree 10 exist? (ii) How many outcomes of rolling 6 identical dice?

Solution. (i) Solutions of $a + b + c = 10$, $a, b, c \geq 0$: $\binom{12}{2} = 66$: the dimension of the degree-10 homogeneous component of $\mathbb{C}[x, y, z]$. In general $\dim \mathbb{C}[x_1 \dots x_k]_n = \binom{n+k-1}{k-1}$ — a fact CSIR asks in algebra clothing. (ii) An outcome of identical dice is a 6-multiset on faces $\{1..6\}$: $\binom{6+6-1}{6} = \binom{11}{6} = 462$. ■

Worked Example 6.3 (*Non-adjacent selections via stars and bars*). Prove: the number of ways to choose r non-consecutive integers from $\{1, 2, \dots, n\}$ is $\binom{n-r+1}{r}$.

Solution. Let the chosen integers be $c_1 < c_2 < \dots < c_r$ and set gaps $g_0 = c_1 - 1$, $g_i = c_{i+1} - c_i - 1$ ($1 \leq i \leq r-1$), $g_r = n - c_r$. The non-consecutive condition says the internal gaps satisfy $g_i \geq 1$; also $g_0, g_r \geq 0$ and $\sum g_i = n - r$. Corollary 6.3 with lower bounds $(0, 1, \dots, 1, 0)$: solutions $= \binom{(n-r)-(r-1)+r}{r} = \binom{n-r+1}{r}$. This gap encoding of increasing sequences is a master bijection: it will reappear for circular versions (Exercise 6), for the Kaplansky lemmas inside the ménage problem, and in partition theory. ■

6.2 Distribution taxonomy: the four classical problems

“Distribute n objects into k boxes” is four problems, not one, according to distinguishability; two are now solved, one is easy, one is deep.

Objects	Boxes	Count of distributions
distinct	distinct	k^n (each object chooses a box)
identical	distinct	$\binom{n+k-1}{k-1}$ (Theorem 6.1)
distinct	identical	$\sum_{j=1}^k \left\{ \begin{matrix} n \\ j \end{matrix} \right\}$ (Stirling numbers; Ch. 11)
identical	identical	$p_{\leq k}(n)$ (partitions; Ch. 18)

The bottom two rows are the gateway to Part II: identifying boxes means quotienting by a symmetric group action that is *not* free, and the elementary tools genuinely stop working. Honest acknowledgment of this boundary — rather than a fake formula — is itself examination-relevant: ISI interviewers probe exactly whether a candidate knows *why* “ $k^n/k!$ ” is wrong for the third row.

The Abstract Viewpoint • Multisets, the H functor, and duality

Write $\binom{n}{k}$ -style notation aside; combinatorialists denote the multiset count by $\left(\begin{matrix} k \\ n \end{matrix} \right) = \binom{n+k-1}{n}$, “ k multichoose n .” Structurally, choosing an n -multiset from $[k]$ is choosing a weakly increasing sequence $1 \leq i_1 \leq \dots \leq i_n \leq k$, and the shift $i_j \mapsto i_j + (j-1)$ is a bijection onto strictly increasing sequences in $[n+k-1]$ — stars and bars is a discrete change of variables between the simplex and the box. In algebraic language: subsets are counted by the exterior power Λ^n , multisets by the symmetric power S^n , and $\dim S^n(\mathbb{C}^k) = \binom{n+k-1}{n}$,

$\dim \Lambda^n(\mathbb{C}^k) = \binom{k}{n}$: the two rows of the selection table are the two classical Schur functors. The identity $\binom{k}{n} = (-1)^n \binom{-k}{n}$ (negative binomial) is the combinatorial reciprocity theorem in its simplest dress: evaluating the polynomial $\binom{x+n-1}{n}$ at negative arguments counts the “opposite” structure. Stanley’s Combinatorial Reciprocity develops this into a general principle relating order polynomials, Ehrhart polynomials of lattice polytopes, and chromatic polynomials — stars and bars is the 1-dimensional Ehrhart theory of the simplex $\{x \geq 0, \sum x_i = n\}$.

6.3 Equations, inequalities, and integer boxes

Worked Example 6.4 (*Inequality to equation*). Count nonnegative integer solutions of $x_1 + x_2 + x_3 \leq 15$.

Solution. Introduce slack $x_4 = 15 - (x_1 + x_2 + x_3) \geq 0$; solutions correspond bijectively to $x_1 + \dots + x_4 = 15$: $\binom{18}{3} = 816$. Slack variables convert simplices into their bounding hyperplane — the same trick as linear programming’s. ■

Worked Example 6.5 (*ISI B.Math pattern*). How many ordered triples of positive integers (a, b, c) satisfy $abc = 720$?

Solution. $720 = 2^4 \cdot 3^2 \cdot 5$. Distribute each prime’s exponent independently among a, b, c : for 2^4 , solutions of $\alpha_1 + \alpha_2 + \alpha_3 = 4$: $\binom{6}{2} = 15$; for 3^2 : $\binom{4}{2} = 6$; for 5 : $\binom{3}{2} = 3$. Product: $15 \cdot 6 \cdot 3 = 270$. Multiplicative structure factors through prime exponents; stars and bars handles each prime. This template covers “number of ways to write N as an ordered product,” “number of divisors” ($k = 1$ box per prime plus one: $d(N) = \prod (\alpha_i + 1)$ is the degenerate case), and CSIR counting-of-ideals questions in $\mathbb{Z}$. ■

Recognition grammar for stars and bars

Trigger phrases: “identical balls/coins/chocolates,” “nonnegative integer solutions,” “at least a_i ,” “number of terms in $(x_1 + \dots + x_k)^n$ ” (answer $\binom{n+k-1}{k-1}$), “dice that are indistinguishable,” “multisets,” “divisor/ordered factorization counts.” Convert: lower bounds by shifting; inequalities by slack; upper bounds $\rightarrow$ flag for inclusion–exclusion; “no box empty” with identical objects $\rightarrow$ positive solutions $\binom{n-1}{k-1}$; “no box empty” with *distinct* objects $\rightarrow$ surjections, Chapter 7, *never* stars and bars.

Exercises

- Count the terms in the expansion of $(a + b + c + d)^{12}$. [EAMCET]
- Find the number of nonnegative integer solutions of $x + y + z = 17$ with x even. [Hint: case or substitute $x = 2t$] [JEE MAIN]
- In how many ways can 12 identical apples and 6 identical oranges be distributed among 4 children? [JEE MAIN]
- Count solutions of $x_1 + x_2 + x_3 + x_4 = 25$, $0 \leq x_i$, with $x_1 \leq 10$ — by complement on the single upper bound. [JEE ADVANCED]

5. Prove that the number of weakly increasing functions $[n] \rightarrow [k]$ is $\binom{n+k-1}{n}$ and the number of strictly increasing ones is $\binom{k}{n}$, via the shift bijection of the Abstract Viewpoint box. [ISI]
6. (Circular non-adjacency; Kaplansky.) Show that the number of ways to choose r pairwise non-adjacent positions from n positions arranged in a *circle* is $\frac{n}{n-r} \binom{n-r}{r}$. [Hint: case on whether position 1 is chosen; use the linear result of the worked example] [OLYMPIAD/ISI]
7. How many ordered pairs (A, B) of multisets on $[k]$ have total size $|A| + |B| = n$? Reconcile the stars-and-bars answer with Vandermonde for multichoose. [CMI]
8. (CSIR flavour.) Show that the number of monomials of degree $\leq n$ in k variables is $\binom{n+k}{k}$, and interpret as the dimension of the space of polynomials of degree $\leq n$. [CSIR]

Inclusion–Exclusion and Derangements

ELI5 • The overcount seesaw

In a class, 20 students like cricket, 15 like chess, and 8 like both. How many like at least one? If you say $20 + 15 = 35$ you have counted the 8 both-likers twice — once in each list. Take them out once: $35 - 8 = 27$. With three hobbies you subtract all the pairwise overlaps, but now the triple-likers, added three times and subtracted three times, have vanished — add them back once. Add, subtract, add, subtract: the seesaw settles exactly when every student has been counted exactly once. That seesaw, written for any number of lists, is inclusion–exclusion.

7.1 The principle

Theorem 7.1 (Inclusion–exclusion). *Let $A_1, \dots, A_k$ be finite subsets of a finite universe U . Then*

$$\left| \bigcup_{i=1}^k A_i \right| = \sum_{\emptyset \neq S \subseteq [k]} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|,$$

equivalently, writing $A_S = \bigcap_{i \in S} A_i$ ($A_\emptyset = U$),

$$\left| U \setminus \bigcup_i A_i \right| = \sum_{S \subseteq [k]} (-1)^{|S|} |A_S|.$$

Proof. Fix $x \in U$ and let $T = \{i : x \in A_i\}$, $t = |T|$. The contribution of x to the right-hand side of the second identity is $\sum_{S \subseteq T} (-1)^{|S|} = \sum_{j=0}^t (-1)^j \binom{t}{j} = (1-1)^t = [t=0]$ by the binomial theorem, where $[\cdot]$ is the Iverson bracket. So each x outside all A_i contributes 1 and every other x contributes 0: the right side counts $U \setminus \bigcup A_i$ exactly. $\square$

The proof deserves contemplation: the entire alternating principle is the single identity $(1-1)^t = 0$ for $t \geq 1$, distributed over elements. This “contribution audit” style of proof — fix an element, total its signed appearances — is the template for Möbius inversion in Chapter 14.

Corollary 7.2 (Symmetric form). *If $|A_S|$ depends only on $|S|$ — say $|A_S| = N_j$ whenever $|S| = j$ — then $|U \setminus \bigcup A_i| = \sum_{j=0}^k (-1)^j \binom{k}{j} N_j$.*

Most examination applications are symmetric, which is why the formula usually appears with binomial coefficients out front.

7.2 Surjections and the distinct-objects/no-empty-box problem

Theorem 7.3 (Surjection count). *The number of surjections from an n -set onto a k -set is*

$$\text{Sur}(n, k) = \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n.$$

Proof. Universe U = all functions $[n] \rightarrow [k]$, $|U| = k^n$. Let A_i be the functions missing the value i . Then $|A_S| = (k - |S|)^n$ (functions into the complement), symmetric in $|S|$; Corollary 7.2 gives the sum, and surjections are exactly the functions avoiding every A_i . $\square$

This settles “distinct objects, distinct boxes, no box empty” — the count the exam-hall temptation writes, wrongly, as stars and bars. Chapter 11 will recognize $\text{Sur}(n, k) = k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

Worked Example 7.1 (*JEE Advanced pattern*). In how many ways can 5 distinct prizes be given to 3 students so that every student gets at least one prize?

Solution. $\text{Sur}(5, 3) = 3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 = 243 - 96 + 3 = 150$. $\blacksquare$

7.3 Derangements

Definition 7.4. A *derangement* of $[n]$ is a permutation with no fixed point. Their number is denoted D_n (also $!n$).

Theorem 7.5. $D_n = n! \sum_{j=0}^n \frac{(-1)^j}{j!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!} \right)$. Moreover D_n is the integer nearest $n!/e$ for $n \geq 1$.

Proof. Let $A_i \subseteq S_n$ be the permutations fixing i . For $|S| = j$, $|A_S| = (n-j)!$ (fix S pointwise, permute the rest freely). Corollary 7.2: $D_n = \sum_j (-1)^j \binom{n}{j} (n-j)! = \sum_j (-1)^j \frac{n!}{j!}$. For the nearest integer claim: $|D_n - n!/e| = n! \left| \sum_{j>n} (-1)^j / j! \right| \leq \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{2}$, the alternating series remainder bound. $\square$

Proposition 7.6 (Recurrences). $D_n = (n-1)(D_{n-1} + D_{n-2})$ for $n \geq 2$, and $D_n = nD_{n-1} + (-1)^n$.

Proof. Combinatorial proof of the first: in a derangement, element 1 maps to some $i \neq 1$ ($n-1$ choices). Case (a): i maps back to 1; the remaining $n-2$ elements form a derangement: D_{n-2} . Case (b): i does not map to 1; then contracting (declare that whatever mapped to $\dots$ precisely: consider the permutation on $[n] \setminus \{1\}$ obtained by replacing the forbidden value 1 in position preimage with forbidden value i ; deranging $[n] \setminus \{1\}$ with the single constraint “ i not fixed” is exactly D_{n-1}). Total $(n-1)(D_{n-2} + D_{n-1})$. The second recurrence follows from the first by induction, or directly from the closed form. $\square$

Worked Example 7.2 (*Partial derangements, JEE Advanced pattern*). Five letters go into five addressed envelopes at random. In how many ways do exactly two letters reach the correct envelope?

Solution. Choose which two are correct: $\binom{5}{2} = 10$; derange the other three: $D_3 = 2$. Answer 20. In general, permutations of $[n]$ with exactly k fixed points number $\binom{n}{k} D_{n-k}$ — the *rencontres numbers*. Summing over k gives the pretty identity $n! = \sum_k \binom{n}{k} D_{n-k}$, which inverts (Chapter 14) back to Theorem 7.5. $\blacksquare$

7.4 Bounded stars and bars

Inclusion–exclusion finishes the business Chapter 6 postponed: upper bounds.

Worked Example 7.3. Count integer solutions of $x_1 + x_2 + x_3 + x_4 = 30$ with $0 \leq x_i \leq 10$.

Solution. Universe: unbounded nonnegative solutions, $\binom{33}{3} = 5456$. Let A_i be solutions with $x_i \geq 11$; by shifting, $|A_i| = \binom{33-11}{3} = \binom{22}{3} = 1540$; $|A_i \cap A_j| = \binom{33-22}{3} = \binom{11}{3} = 165$; triple intersections need x -total $\geq 33 > 30$: empty. Answer $5456 - 4 \cdot 1540 + 6 \cdot 165 = 5456 - 6160 + 990 = 286$. As a check, by symmetry $x_i \mapsto 10 - x_i$ this equals the count for total 10: $\binom{13}{3} = 286$. Symmetric sanity checks of this kind are exam-hall gold. ■

The same computation, read through generating functions, is the coefficient of x^{30} in $(1 + x + \dots + x^{10})^4 = \left(\frac{1-x^{11}}{1-x}\right)^4$ — Chapter 12 will make inclusion–exclusion and the algebra of $(1 - x^{11})^4$ literally the same calculation.

7.5 Euler’s totient and arithmetic applications

Theorem 7.7. For $n = p_1^{a_1} \cdots p_r^{a_r}$, the number of integers in $[n]$ coprime to n is $\varphi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$.

Proof. Let $A_i = \{m \in [n] : p_i \mid m\}$, so $|A_S| = n / \prod_{i \in S} p_i$ exactly (each $p_i \mid n$). Inclusion–exclusion: $\varphi(n) = \sum_S (-1)^{|S|} n / \prod_{i \in S} p_i = n \prod_i (1 - 1/p_i)$, expanding the product. □

The expansion step — recognizing an inclusion–exclusion sum as an expanded product — is a two-way street: products of $(1 - \text{something})$ factors *are* inclusion–exclusions. This viewpoint, systematized, is Möbius inversion over the divisor lattice (Chapter 14), of which Theorem 7.7 is the flagship instance.

The Abstract Viewpoint • Inclusion–exclusion is Möbius inversion on the Boolean lattice

Define, for $S \subseteq [k]$, $f(S)$ = number of elements lying in *exactly* the sets indexed by S , and $g(S) = |A_S|$ = number lying in *at least* those. Then $g(S) = \sum_{T \supseteq S} f(T)$, and Theorem 7.1 is the inversion $f(S) = \sum_{T \supseteq S} (-1)^{|T|-|S|} g(T)$: the Möbius function of the Boolean lattice is $\mu(S, T) = (-1)^{|T \setminus S|}$. Rota’s insight (1964) was that *every* locally finite poset carries such a μ , with the divisor lattice giving the number-theoretic Möbius function and the partition lattice giving the formulas behind free cumulants. The sign-alternating sums of this chapter are thus one column of a general theory of “difference operators on posets”; Chapter 14 builds it, and CSIR/NBHM increasingly examine it. A probabilistic sibling: truncating the alternating sum gives the Bonferroni inequalities — partial sums alternately over- and under-estimate — which is how one uses inclusion–exclusion when exact intersection data is unavailable.

Inclusion–exclusion under time pressure

(1) Identify the “bad properties” A_i ; the answer wanted is usually $|U| - |\bigcup A_i|$ (avoid all) — write the complemented form directly. (2) Verify the symmetry $|A_S| = N_{|S|}$ before using binomial prefactors; asymmetric bounds (different b_i) force the full subset sum. (3) Truncate when intersections empty out (as in the bounded-sum example) — most exam instances die after two or three terms. (4) Landmark counts to memorize: $D_4 = 9$, $D_5 = 44$, $D_6 = 265$;

$\text{Sur}(n, 2) = 2^n - 2$; $\text{Sur}(n, 3) = 3^n - 3 \cdot 2^n + 3$. (5) The *ménage* problem, couples-not-adjacent problems, and “no two of these k people adjacent at a circular table with extra structure” all reduce to inclusion–exclusion over which forbidden adjacencies occur, counted with the circular gap lemma (Exercise 6.6).

Exercises

1. How many integers in $[1000]$ are divisible by none of 2, 3, 5? [\[EAMCET/JEE MAIN\]](#)
2. How many arrangements of UNIVERSE contain none of the blocks UNI, VERSE? (Blocks as consecutive substrings.) [\[JEE ADVANCED\]](#)
3. Prove $\sum_{k=0}^n \binom{n}{k} D_{n-k} = n!$ combinatorially, and invert it formally to re-derive D_n assuming the binomial inversion formula $f(n) = \sum_k \binom{n}{k} g(k) \iff g(n) = \sum_k (-1)^{n-k} \binom{n}{k} f(k)$. [\[ISI/CMI\]](#)
4. Count the permutations of $[8]$ with no fixed point and no 2-cycle containing both 1 and 2... simpler: count permutations of $[8]$ fixing none of 1, 2, 3 (only these three positions restricted). [\[JEE ADVANCED\]](#)
5. Solve the postponed problem: n couples in a row of $2n$ seats, no couple adjacent. Show the answer is $\sum_{k=0}^n (-1)^k \binom{n}{k} 2^k (2n-k)!$. [\[ISI/OLYMPIAD\]](#)
6. (*Ménage*, guided.) Alternating circular seating of n couples with no couple adjacent: seat the women first ($2 \cdot n!$ ways with labelled chairs up to the standard normalization), then show the men’s count is $M_n = \sum_k (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!$ using the circular non-adjacency lemma. [\[OLYMPIAD\]](#)
7. Derive the Bonferroni inequality: the partial sums of Theorem 7.1 truncated after an even number of terms underestimate $|\bigcup A_i|$ and after an odd number overestimate. *[Hint: audit each element’s partial contribution $\sum_{j \leq m} (-1)^{j+1} \binom{t}{j}$ via Theorem 5.9]* [\[CSIR/NBHM\]](#)
8. Show that the probability that a uniformly random permutation of a large deck has no fixed card tends to $1/e$, and compute the limiting probability of exactly k fixed cards (Poisson(1)). [\[CSIR\]](#)

The Arithmetic of Factorials: Legendre, Kummer, Lucas

ELI5 • How many zeros end 100!?

100! is a gigantic number ending in a long tail of zeros. Each zero is a factor of $10 = 2 \times 5$, and among $1, \dots, 100$ the factor 5 is much rarer than the factor 2 — so count the 5s. The multiples of 5 up to 100 donate twenty 5s; the multiples of 25 donate one *extra* 5 each (four more); 125 exceeds 100, so stop. Twenty-four 5s, plenty of 2s: exactly 24 zeros. The trick — count multiples of p , then of p^2 , then $p^3, \dots$ — works for any prime and any factorial.

8.1 Legendre's formula

For a prime p and $n \geq 1$ let $v_p(n)$ denote the exponent of p in n (the p -adic valuation), so $v_p(ab) = v_p(a) + v_p(b)$.

Theorem 8.1 (Legendre, 1808). $v_p(n!) = \sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - s_p(n)}{p - 1}$, where $s_p(n)$ is the digit sum of n in base p .

Proof. For the sum: count pairs (m, i) with $m \leq n$ and $p^i \mid m$. Summed over m first, each m contributes $v_p(m)$ pairs, totalling $v_p(n!)$; summed over i first, level i contributes $\lfloor n/p^i \rfloor$ multiples. Double counting equates them. For the digit-sum form: write $n = \sum_j a_j p^j$ in base p ; then $\lfloor n/p^i \rfloor = \sum_{j \geq i} a_j p^{j-i}$, and

$$\sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_j a_j \sum_{i=1}^j p^{j-i} = \sum_j a_j \frac{p^j - 1}{p - 1} = \frac{n - s_p(n)}{p - 1}.$$

□

Worked Example 8.1 (*JEE Main pattern*). Find the exponent of 3 in 100!, and the number of trailing zeros of 1000!.

Solution. $v_3(100!) = \lfloor 100/3 \rfloor + \lfloor 100/9 \rfloor + \lfloor 100/27 \rfloor + \lfloor 100/81 \rfloor = 33 + 11 + 3 + 1 = 48$. Trailing zeros of 1000!: $v_5 = 200 + 40 + 8 + 1 = 249$. ■

8.2 Kummer's theorem: carries count

Theorem 8.2 (Kummer, 1852). For $0 \leq r \leq n$, the valuation $v_p\binom{n}{r}$ equals the number of carries when r is added to $n - r$ in base p .

Proof. Write $m = n - r$. By Legendre's digit-sum form,

$$v_p\binom{n}{r} = v_p(n!) - v_p(r!) - v_p(m!) = \frac{(s_p(r) + s_p(m) - s_p(n))}{p-1}.$$

Adding $r + m$ in base p , each carry converts a column sum of at least p into a digit reduced by p and a 1 passed leftward, decreasing the total digit sum by exactly $p - 1$. Hence $s_p(r) + s_p(m) - s_p(r + m) = (p - 1) \cdot (\# \text{carries})$. $\square$

Corollary 8.3. $\binom{n}{r}$ is odd iff the binary representation of r is a submask of that of n (every binary digit of r is $\leq$ the corresponding digit of n). Consequently the number of odd entries in row n of Pascal's triangle is $2^{s_2(n)}$.

Proof. Oddness means zero carries in binary addition of r and $n - r$, i.e. no column where both addends have a 1 — exactly the submask condition. Each 1-digit of n leaves a free binary choice for r 's digit; each 0-digit forces 0. $\square$

This corollary is the arithmetic engine of the Sierpiński-triangle pattern in Pascal's triangle mod 2, a fixture of ISI subjective papers and olympiad folklore.

8.3 Lucas's theorem

Theorem 8.4 (Lucas, 1878). Let p be prime and write $n = \sum n_i p^i$, $r = \sum r_i p^i$ in base p . Then

$$\binom{n}{r} \equiv \prod_i \binom{n_i}{r_i} \pmod{p},$$

with the convention $\binom{n_i}{r_i} = 0$ if $r_i > n_i$.

Proof. In $\mathbb{F}_p[x]$, the Frobenius/freshman identity $(1+x)^p \equiv 1+x^p$ holds since the middle binomials $\binom{p}{j}$, $0 < j < p$, are divisible by p (their numerator $p!$ contains p ; the denominator $j!(p-j)!$ does not). Iterating, $(1+x)^{p^i} \equiv 1+x^{p^i}$, hence

$$(1+x)^n = \prod_i ((1+x)^{p^i})^{n_i} \equiv \prod_i (1+x^{p^i})^{n_i} \pmod{p}.$$

Extract the coefficient of x^r : since base- p digits are unique, the only way to assemble x^r from the product is to take $x^{r_i p^i}$ from the i -th factor, contributing $\binom{n_i}{r_i}$. $\square$

Worked Example 8.2 (Olympiad/CSIR pattern). Compute $\binom{1000}{300} \pmod{7}$, and determine whether $\binom{2026}{1013}$ is even.

Solution. Base 7: $1000 = 2626_7$ (since $2 \cdot 343 + 6 \cdot 49 + 2 \cdot 7 + 6 = 1000$), $300 = 606_7 = 0 \cdot 343 + 6 \cdot 49 + 0 \cdot 7 + 6$. Lucas: $\binom{2}{0} \binom{6}{6} \binom{2}{0} \binom{6}{6} = 1 \cdot 1 \cdot 1 \cdot 1 = 1 \pmod{7}$. Parity of $\binom{2026}{1013}$: $2026 = 1111101010_2$, $1013 = 111110101_2$; the addition $1013 + 1013 = 2026$ in binary certainly carries (any addition of a number to itself with a nonzero digit carries), so by Kummer the valuation is positive: even. (In general $\binom{2n}{n}$ is even for $n \geq 1$: base-2 addition $n + n$ always carries.) $\blacksquare$

8.4 Integrality and divisibility via counting

- Proposition 8.5.** (a) $\frac{(mn)!}{(m!)^n n!} \in \mathbb{Z}$: it counts partitions of an mn -set into n unlabeled blocks of size m .
- (b) $\frac{(2n)!}{2^n n!} = 1 \cdot 3 \cdot 5 \cdots (2n-1) \in \mathbb{Z}$: it counts perfect matchings of a $2n$ -set.
- (c) $n!$ divides the product of any n consecutive integers.
- (d) $\binom{2n}{n}$ divides $\text{lcm}(1, 2, \dots, 2n)$ — more precisely $\binom{2n}{n} \mid \text{lcm}(1, \dots, 2n)$, since for each prime p , $v_p\left(\binom{2n}{n}\right) \leq \log_p(2n)$ by Kummer (carries occur only in the first $\lfloor \log_p 2n \rfloor$ columns).

Proof. (a) Distribute mn labelled objects into n labelled boxes of size m : multinomial $\frac{(mn)!}{(m!)^n}$; unlabel the boxes: the S_n -action on labelled block-structures is free (blocks have distinct contents), divide by $n!$. (b) is (a) with $m = 2$ rewritten, or Exercise 1.6. (c) The product of n consecutive integers starting at $m+1$ is $(m+n)^{\underline{n}} = \binom{m+n}{n} n!$. (d) as indicated: every prime power dividing $\binom{2n}{n}$ is at most $2n$. $\square$

The moral is a proof pattern of high examination value: *to prove an expression is an integer, or a divisibility, exhibit it as counting something*. Purely arithmetic proofs via Legendre are available (and worth writing once as an exercise), but the combinatorial certificates are shorter, more robust, and are what ISI/CMI subjective graders hope to see.

The Abstract Viewpoint • p -adic continuity of factorials

Legendre's formula says $v_p(n!) \approx n/(p-1)$: factorials are highly divisible, and $1/n!$ blows up p -adically. Yet binomial coefficients — ratios of factorials — remain integers, hence p -adically bounded; Lucas says more: the function $n \mapsto \binom{n}{r} \bmod p$ is determined digitwise, i.e. it factors through the digit expansion, a strong self-similarity that draws the Sierpiński gasket. In modern language, the binomial coefficient polynomials $\binom{x}{r}$ form an orthonormal basis of the space of continuous functions $\mathbb{Z}_p \rightarrow \mathbb{Q}_p$ (Mahler's theorem): every continuous p -adic function expands uniquely as $\sum a_r \binom{x}{r}$ with $a_r \rightarrow 0$. Thus the finite-difference calculus of Chapter 5's abstract box is, p -adically, the *whole* of analysis. Granville, and more recently the study of digit functions in analytic number theory, extend Kummer to prime powers; and the same carry-counting reappears in the theory of base- p representations of algebraic groups (Steinberg tensor product theorem), which CSIR aspirants in algebra will one day recognize as Lucas's theorem for representations.

Exam radar

Trailing zeros $\rightarrow v_5$; “largest k with $p^k \mid n!$ ” $\rightarrow$ Legendre; “is $\binom{n}{r}$ divisible by p ” $\rightarrow$ Lucas digit check (any digit of r exceeding n 's digit $\Rightarrow$ yes); parity of binomial coefficients $\rightarrow$ submask rule; “number of odd entries in a row” $\rightarrow 2^{s_2(n)}$; “show this ratio is an integer” $\rightarrow$ find what it counts. Numbers to know: $v_5(100!) = 24$, $v_2(100!) = 97$, $\binom{2n}{n}$ even for $n \geq 1$, $\binom{p}{j} \equiv 0 \pmod{p}$ for $0 < j < p$, and $\binom{p^a}{j} \equiv 0 \pmod{p}$ likewise.

Exercises

- Find the number of trailing zeros of $2026!$ and the exponent of 7 in $\binom{2026}{1000}$ via Kummer. [JEE/CSIR]

2. Prove $\binom{p^a-1}{r} \equiv (-1)^{s_p(r)} \pmod{p}$ for $0 \leq r < p^a$. [Hint: every base- p digit of $p^a - 1$ is $p - 1$; Lucas plus $\binom{p-1}{j} \equiv (-1)^j$] [\[OLYMPIAD\]](#)
3. Show that $\binom{2n}{n} \equiv 2 \pmod{4}$ iff n is a power of 2. [Hint: Kummer: exactly one carry] [\[OLYMPIAD/ISI\]](#)
4. Prove part (c) of Proposition 8.5 purely arithmetically: show $v_p(n!) \leq v_p((m+1) \cdots (m+n))$ for all p using $\lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$. [\[ISI/CMI\]](#)
5. How many $r \in \{0, 1, \dots, 2^{10}\}$ make $\binom{2^{10}}{r}$ odd? [\[ISI\]](#)
6. Prove that the product of k consecutive odd integers is divisible by nothing forced... replace with: prove $\frac{(3n)!}{(3!)^n n!}$ is an integer and interpret it. [\[CMI\]](#)
7. (Erdős calibration.) Using $\binom{2n}{n} \geq 4^n/(2n+1)$ and $\binom{2n}{n} \mid \text{lcm}(1, \dots, 2n)$, deduce $\text{lcm}(1, \dots, m) \geq 2^{m-2}$ for $m \geq 7$ — a Chebyshev-type bound by pure counting. [\[NBH-M/OLYMPIAD\]](#)

Lattice Paths, Reflection, and the Catalan Numbers

ELI5 • City-block walks

You stand at a street corner and must walk 4 blocks east and 3 blocks north to reach a café, never wasting a step. Your walk is a word like EENENNE: seven letters, four E's, three N's. Choosing the walk is choosing which 3 of the 7 letters are N: $\binom{7}{3} = 35$ walks. Every “shortest routes on a grid” problem is secretly a spelling problem.

9.1 Unrestricted paths and path identities

Theorem 9.1. *The number of monotone lattice paths from $(0,0)$ to (m,n) using unit steps $E = (1,0)$ and $N = (0,1)$ is $\binom{m+n}{n}$.*

Proof. The path is the word in $\{E, N\}$ recording its steps: a multiset permutation of m E's and n N's — Theorem 3.5. $\square$

Path pictures convert binomial identities into geometry: Pascal's rule is “the last step was E or N”; Vandermonde is “cut the path where it crosses a vertical line”; the hockey stick is “where does the path last leave the bottom row.” Training the eye to redraw an identity as a path statement is among the highest-yield skills for ISI and olympiad work.

Worked Example 9.1 (*Paths through and avoiding a point*). Of the paths from $(0,0)$ to $(8,8)$, how many pass through $(3,5)$? How many avoid it?

Solution. Through: $\binom{8}{3}\binom{8}{5} = 56 \cdot 56 = 3136$ (concatenate the two legs; product rule). Avoid: $\binom{16}{8} - 3136 = 12870 - 3136 = 9734$. $\blacksquare$

9.2 The reflection principle

Theorem 9.2 (Ballot problem; Bertrand 1887, reflection of D. André). *In an election, candidate A receives a votes and B receives b votes, $a > b$. The number of orders of counting in which A stays strictly ahead of B throughout is $\frac{a-b}{a+b}\binom{a+b}{a}$.*

Proof. Model tallies as paths: step $+1$ per A-vote, -1 per B-vote; a counting order is a path from $(0,0)$ to $(a+b, a-b)$, and “strictly ahead throughout” means the path stays strictly above the axis after time 0. Any such path must start with an A-step, reaching $(1,1)$. Count instead the bad paths from $(1,1)$ to $(a+b, a-b)$ that touch the axis: reflect the segment up to the first axis-touch across the axis, obtaining a path from $(1,-1)$ to $(a+b, a-b)$; reflection is an involution

between bad paths and *all* paths from $(1, -1)$ (any path from $(1, -1)$ to a positive height must cross the axis, so all are images). Hence

$$\#_{\text{good}} = \binom{a+b-1}{a-1} - \binom{a+b-1}{a} = \frac{a-b}{a+b} \binom{a+b}{a},$$

the last step by the factorial forms. $\square$

Reflection is the third great transform of this book (after gluing and gaps): it converts a *barrier* condition into a difference of two free counts. It powers the Catalan formula next, the cycle lemma, and (in Part II) the Lindström–Gessel–Viennot determinant for non-crossing path families.

9.3 Catalan numbers

Definition 9.3. The n -th *Catalan number* is $C_n = \frac{1}{n+1} \binom{2n}{n}$: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...

Theorem 9.4 (Dyck paths). C_n counts the lattice paths from $(0, 0)$ to $(2n, 0)$ with steps $U = (1, 1)$, $D = (1, -1)$ that never go below the axis (Dyck paths); equivalently, balanced strings of n pairs of parentheses in which every prefix has at least as many ‘(’ as ‘)’.

Proof. Free paths to $(2n, 0)$: $\binom{2n}{n}$. Bad paths touch level -1 ; reflect at the first touch to get an arbitrary path from $(0, -2)$ to $(2n, 0)$, i.e. with $n+1$ up and $n-1$ down steps: $\binom{2n}{n-1}$ of them, and again every such path arises (it must cross level -1). Hence $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$ by direct factorial algebra. $\square$

Theorem 9.5 (The Catalan recurrence and its universality). $C_0 = 1$ and $C_{n+1} = \sum_{k=0}^n C_k C_{n-k}$. Any family of structures satisfying this “first-return decomposition” is counted by Catalan numbers. Among such families:

- (i) triangulations of a convex $(n+2)$ -gon;
- (ii) binary trees with n internal nodes; full parenthesizations of a product of $n+1$ factors;
- (iii) non-crossing perfect matchings of $2n$ points on a circle;
- (iv) non-crossing partitions of $[n]$;
- (v) sequences $1 \leq a_1 \leq \dots \leq a_n$ with $a_i \leq i$;
- (vi) standard Young tableaux of shape $2 \times n$;
- (vii) permutations of $[n]$ avoiding the pattern 123 (or any fixed pattern of length 3).

Proof of the recurrence. A nonempty Dyck path factors uniquely as $UPDQ$ where P is the (translated) Dyck path strictly above the axis before the first return to height 0, and Q is the Dyck path after it. If the whole path has length $2(n+1)$ and P has semilength k , then Q has semilength $n-k$; sum over k . For the listed families one exhibits first-return-style decompositions or direct bijections; we prove (ii) and (vii) as worked examples below and leave the rest as guided exercises — learning three of these bijections cold is worth more than memorizing thirty formulas. $\square$

Worked Example 9.2 (Binary trees $\leftrightarrow$ Dyck paths). A binary tree is empty or a root with an ordered pair of binary subtrees. Encode: empty tree $\mapsto$ empty word; tree with subtrees

$(L, R) \mapsto U \langle L \rangle D \langle R \rangle$. Show this is a bijection with Dyck paths, semilength = number of internal nodes.

Solution. The map is well defined by structural induction and lands in Dyck words (the U opened at the root is matched by its D after $\langle L \rangle$, which is nonnegative inductively). Conversely the first-return factorization $U P D Q$ of a nonempty Dyck word uniquely names L (decode P) and R (decode Q); induction on length gives mutual inverses. Hence binary trees with n nodes are equinumerous with Dyck paths of semilength n : C_n . The same grammar $T = \varepsilon \mid U T D T$ is the algebraic reason the generating function of Chapter 12 satisfies $C(x) = 1 + xC(x)^2$. ■

Worked Example 9.3 (123-avoiding permutations, ISI/CMI pattern). Show that permutations of $[n]$ with no increasing subsequence of length 3 number C_n .

Solution. Sketch of the standard bijection: scan σ left to right, maintaining the sequence of left-to-right minima; σ avoids 123 iff the non-minima form a decreasing sequence, and the positions/values data of the minima form a Dyck-path shadow. Concretely, record at step i a U if $\sigma(i)$ is a new left-to-right minimum and a D otherwise, then correct end-effects... the clean formal proof pairs σ with the pair of lattice words given by (positions of minima, values of minima) and checks the ballot condition. We flag honestly: this bijection has details; the reader should either work them fully (excellent training) or take the slicker route through the RSK correspondence, where avoidance of 123 means both tableaux have at most 2 rows, and shape- $2 \times$ counting is (vi). Either way the count is C_n . ■

Worked Example 9.4 (JEE Advanced adjacency). In how many ways can $2n$ people seated around a table shake hands in n simultaneous pairs with no two arms crossing?

Solution. This is family (iii): C_n . For $n = 5$: $C_5 = 42$. ■

The Abstract Viewpoint • Why $\frac{1}{n+1}$: the cycle lemma and quotients

The mysterious factor $\frac{1}{n+1}$ has a symmetry explanation (Dvoretzky–Motzkin cycle lemma): among the $\binom{2n+1}{n}$ arrangements of $n+1$ U's and n D's on a *circle* of $2n+1$ steps, each circular class of the $2n+1$ rotations contains *exactly one* rotation that is a “dominating” sequence; hence the count of dominating linear sequences is $\frac{1}{2n+1} \binom{2n+1}{n} = C_n$. The Catalan numbers are thus an orbit count for a free $\mathbb{Z}/(2n+1)$ -action — Chapter 4's philosophy vindicated. Deeper: C_n enumerates the non-crossing partition lattice $NC(n)$, whose Möbius-theoretic structure underlies free probability (free cumulants — Speicher), while the full partition lattice underlies classical cumulants; the passage classical $\rightarrow$ free is precisely crossing $\rightarrow$ non-crossing. And the Catalan generating function $C(x) = \frac{1 - \sqrt{1-4x}}{2x}$, an algebraic function, exemplifies the meta-theorem that context-free grammars have algebraic generating functions (Chomsky–Schützenberger) — the grammar being the $T = \varepsilon \mid U T D T$ of the worked example.

Catalan radar

Any of these phrases signals Catalan: balanced brackets; mountain ranges; ways to triangulate; ways to associate/bracket a product; non-crossing chords/ handshakes; ballot-type “never behind” conditions; queues at a counter with Rs. 50 and Rs. 100 notes and no change to start (classic JEE Advanced 2000-era problem: answer C_n arrangements times orderings if people are distinct — read carefully whether people are distinguishable!); paths below the diagonal; stack-sortable sequences. If the count starts $1, 1, 2, 5, 14, \dots$, conjecture Catalan and seek a first-return decomposition or a reflection.

Exercises

1. How many shortest paths from $(0, 0)$ to $(7, 5)$ pass through neither $(3, 3)$ nor $(5, 2)$? [JEE MAIN]
2. $2n$ people, of whom n hold Rs. 50 notes and n hold Rs. 100 notes, queue at a counter selling Rs. 50 tickets with an empty till. If the note-holders are *indistinguishable* within each type, show the number of queue orders in which nobody waits for change is C_n ; if all $2n$ people are distinct, show it is $C_n \cdot n! \cdot n!$. [JEE ADVANCED]
3. Prove bijectively that triangulations of a convex $(n+2)$ -gon are counted by binary trees with n internal nodes (duality: triangle $\mapsto$ node). [ISI/CMI]
4. Prove family (v): sequences $1 \leq a_1 \leq \dots \leq a_n$, $a_i \leq i$, are counted by C_n , by reading a Dyck path's heights. [ISI]
5. Show that the number of paths from $(0, 0)$ to (n, n) that stay weakly below the diagonal and touch it exactly $k+1$ times (including endpoints) is $\frac{k}{2n-k} \binom{2n-k}{n} \dots$ if this resists, prove instead the first-return refinement: Dyck paths of semilength n with first return at $2k$ number $C_{k-1}C_{n-k}$. [OLYMPIAD]
6. Use the reflection principle to count paths from $(0, 0)$ to (m, n) , $m \geq n$, that never rise above the diagonal $y = x$: $\frac{m-n+1}{m+1} \binom{m+n}{n}$ (ballot numbers). [ISI/OLYMPIAD]
7. Prove $\sum_{k=0}^n C_k C_{n-k} = C_{n+1}$ directly from the Dyck-path factorization, and verify numerically through $n = 4$. [JEE ADVANCED]
8. (NBHM.) Show C_n is odd iff $n = 2^k - 1$ for some k . [Hint: Kummer on $\binom{2n}{n}$ and the factor $n+1$] [NBHM/OLYMPIAD]

The Examination Hall: Strategy, Traps, and Worked Sets

This closing chapter of Part I is unapologetically practical. The theory of Chapters 1–9 is a workshop of tools; here we drill tool *selection* — the skill examinations actually price — and walk through calibrated problem sets in the style of each target examination.

10.1 The master decision procedure

Seven questions before any computation

1. **What is being counted?** Write the set in one clean sentence. If you cannot, the problem is not yet understood.
2. **Order? Repetition? Distinguishability?** Locate the cell of the selection table / distribution taxonomy.
3. **Geometry?** Row, circle, labelled seats? Anchor circles once.
4. **Constraints?** Classify each: adjacency (glue/gap/complement), position pins (fill first), bounds (shift; upper bounds $\rightarrow$ PIE), forbidden patterns (complement or reflection), “at least one” (complement), “no box empty” (positive stars-and-bars if identical; surjections if distinct).
5. **One structure or a chain?** Chains multiply; alternatives add. Guard the product rule’s two hypotheses.
6. **Small-case sanity.** Compute $n = 1, 2, 3$ by hand; match your formula. Thirty seconds here refunds minutes of confidence.
7. **Order-of-magnitude check.** Answers must respect obvious bounds (at most $n!$, at least the unconstrained relaxation, integrality after division).

10.2 The trap museum

Each exhibit is a genuine, repeatedly observed examination error.

Trap 1: Dividing by $r!$ with repeated objects

Selecting 2 letters from $\{A, A, B\}$: ordered selections are AA, AB, BA — three, not $3 \cdot 2$; and unordered selections are $\{A, A\}, \{A, B\}$: two, not $3/2$. Division needs uniform fibres (Lemma 2.3); repetition breaks uniformity. Correct tool: casework on the multiplicity pattern, or generating functions (Chapter 12).

Trap 2: “At least” by pre-placing

“Distribute 10 distinct toys to 3 children, each child at least one”: the tempting $\binom{10}{1}\binom{9}{1}\binom{8}{1} \cdot 3^7$ overcounts massively (a distribution where child 1 holds toys $\{1, 2\}$ is generated multiple times depending on which toy was the “guaranteed” one). Correct: surjections, $\text{Sur}(10, 3) = 3^{10} - 3 \cdot 2^{10} + 3 = 55980$.

Trap 3: Double-anchoring circles

Alternating men/women at a round table: anchor *once* (seat men in $(n-1)!$), then women linearly in the n gaps $(n!)$ — not $(n-1)!(n-1)!$.

Trap 4: Complement of the wrong universe

“Words from EQUATION with no two vowels adjacent”: the complement “some two vowels adjacent” is a nightmare of overlapping events; the gap method is direct and clean. Complement is for *at least one*-type constraints with simple bad sets, not for adjacency webs.

Trap 5: Confusing “exactly” with “at least” in PIE

Inclusion–exclusion’s basic output is *none of the properties* (or, by complement, *at least one*). “Exactly m properties” needs the generalized PIE:

$$E_m = \sum_{j \geq m} (-1)^{j-m} \binom{j}{m} N_j, \quad N_j = \sum_{|S|=j} |A_S|,$$

provable by the same contribution audit as Theorem 7.1 (an element with exactly t properties contributes $\sum_j (-1)^{j-m} \binom{j}{m} \binom{t}{j} = [t = m]$).

Trap 6: Leading zeros and boundary digits

Every “how many numbers” problem must quarantine the leading digit. Cases on whether 0 is used are usually unavoidable; do them first, not as a patch.

Trap 7: Identical boxes handled by division

Distributing 6 distinct books into 3 identical parcels of 2: $\frac{1}{3!} \binom{6}{2} \binom{4}{2} \binom{2}{2} = 15$ is *correct* (equal block sizes $\Rightarrow$ free action), but into parcels of sizes 1, 2, 3 the division by $3!$ is *wrong* (blocks are already distinguishable by size; no division). Mixed equal sizes: divide by $\prod_j (\text{count of blocks of size } j)!$ only. The general principle: divide by the symmetries that genuinely act freely, nothing more.

10.3 Worked set A: EAMCET / JEE Main tempo

Worked Example 10.1. How many 4-letter words (meaningful or not) can be formed from the letters of LOGARITHM, no letter repeated?

Solution. 9 distinct letters: $9^4 = 3024$. ■

Worked Example 10.2. A bag has 5 red, 4 white, 3 blue balls, all distinct. How many selections of 3 balls contain at least one ball of each colour? At least one red?

Solution. Each colour: $5 \cdot 4 \cdot 3 = 60$. At least one red: $\binom{12}{3} - \binom{7}{3} = 220 - 35 = 185$. ■

Worked Example 10.3. Number of diagonals of a convex n -gon; and for $n = 12$, the number of intersection points of diagonals strictly inside, no three concurrent.

Solution. Diagonals: $\binom{n}{2} - n = \frac{n(n-3)}{2}$; for $n = 12$: 54. Interior intersections: each is determined by a unique 4-subset of vertices (the two crossing diagonals of that quadrilateral): $\binom{12}{4} = 495$. The second count is a pure bijection-principle showpiece. ■

Worked Example 10.4. How many 5-digit numbers divisible by 3 use only digits $\{0, 1, 2, 3, 4, 5\}$ without repetition?

Solution. Digit sum $0 + 1 + \dots + 5 = 15$; a 5-subset omits one digit, and divisibility by 3 of the sum requires omitting 0 or 3. Omit 0: digits $\{1, 2, 3, 4, 5\}$, arrangements $5! = 120$. Omit 3: digits $\{0, 1, 2, 4, 5\}$: $5! - 4! = 96$ (subtract leading zero). Total 216. ■

10.4 Worked set B: JEE Advanced texture

Worked Example 10.5. Let N be the number of ways to place the integers 1..9 in a 3×3 array so that every row increases left to right and every column increases top to bottom. Find N .

Solution. These are standard Young tableaux of shape 3×3 . Elementary route: 1 sits top-left and 9 bottom-right; case on the position of 2 and use symmetry — feasible but bushy. The hook length formula (stated here, proved in the literature; its statement is fair game for advanced aspirants) gives $N = \frac{9!}{\prod \text{hooks}} = \frac{362880}{5 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1} = \frac{362880}{8640} = 42 = C_4 \cdot 3$? Compute: $5 \cdot 4 \cdot 3 = 60$, $4 \cdot 3 \cdot 2 = 24$, $3 \cdot 2 \cdot 1 = 6$; $60 \cdot 24 \cdot 6 = 8640$; $N = 42$. (Coincidentally $C_5 = 42$: shape 2×5 would give exactly Catalan; shape 3×3 gives 42 by hooks.) ■

Worked Example 10.6. Find the number of 7-digit numbers whose digits are nondecreasing.

Solution. Nondecreasing strings of length 7 over $\{1, \dots, 9\}$ (a leading 0 would force all zeros) are 7-multisets of $\{1..9\}$: $\binom{15}{7}$? Careful: $\binom{7+9-1}{7} = \binom{15}{7} = 6435$. Digits over $\{0, \dots, 9\}$ nondecreasing with nonzero lead are the same as multisets over $\{1..9\}$ since a nondecreasing string starting nonzero never contains 0. Answer 6435. ■

Worked Example 10.7. In how many ways can three numbers in arithmetic progression be selected from $\{1, 2, \dots, 2n\}$?

Solution. An AP $(a, a + d, a + 2d)$, $d \geq 1$, needs $a + 2d \leq 2n$: for each $d \leq n - 1 \dots$ precisely d ranges 1.. $n - 1$? Compute: $a \leq 2n - 2d$, so $\sum_{d \geq 1} \max(2n - 2d, 0) = 2(n - 1) + 2(n - 2) + \dots + 2 = 2 \cdot \frac{(n-1)n}{2} = n(n - 1)$. Equivalently: choose the outer pair $\{x, y\}$ with $x \equiv y \pmod{2}$ (the middle term must be integral): same-parity pairs number $2\binom{n}{2} = n(n - 1)$. Two roads, one count — the second is the graders' favourite. ■

10.5 Worked set C: ISI / CMI / IAT subjective texture

Worked Example 10.8. Prove that among any $n + 1$ integers chosen from $[2n]$, some two are coprime, and some one divides another.

Solution. Coprime: among $n + 1$ numbers in $[2n]$, two are consecutive (pigeonhole on the n pairs $\{2k - 1, 2k\}$), and consecutive integers are coprime. Divisibility: write each chosen $m = 2^a q$ with

q odd; the odd parts live among the n odd numbers in $[2n]$, so two chosen numbers share an odd part and the one with smaller 2-power divides the other. Both arguments are pigeonhole with a *designed* partition — Chapter 15 makes this a method. ■

Worked Example 10.9. For $n \geq 1$ show $\sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}^{-1}$ has no nice closed form — replace by the honest task: show $\sum_k \binom{n}{k} 2^k = 3^n$ combinatorially and generalize.

Solution. 3^n counts functions $[n] \rightarrow \{0, 1, 2\}$; classify by the preimage S of $\{1, 2\}$: choose S ($\binom{n}{|S|}$ summed as $\binom{n}{k}$) then a 2-colouring of S (2^k). Generally $\sum_k \binom{n}{k} t^k = (1+t)^n$ counts $[n] \rightarrow \{0, 1, \dots, t\}$ -style structures for integral t by the identical stratification — the binomial theorem proved by counting, for every positive integer substitution, hence as a polynomial identity. ■

Worked Example 10.10. (IAT/ISI objective.) The number of onto functions $[6] \rightarrow [3]$ whose fibres have sizes 3, 2, 1 in some order.

Solution. Choose the fibre-size assignment to the 3 range elements: $3! = 6$ orders; for the order $(3, 2, 1)$: $\binom{6}{3} \binom{3}{2} \binom{1}{1} = 20 \cdot 3 = 60$. Total 360. Check against all surjections: $\text{Sur}(6, 3) = 3^6 - 3 \cdot 2^6 + 3 = 540$; the remaining $540 - 360 = 180$ have fibre profile $(2, 2, 2)$: $\frac{6!}{(2!)^3} \cdot \frac{3!}{3!} = 90$ — verify: ordered blocks $\binom{6}{2} \binom{4}{2} \binom{2}{2} = 90$, assigned to labelled range elements as-is (blocks already assigned in the choosing): 90? That undercounts: choosing fibres *for* range elements 1, 2, 3 in order is exactly $\binom{6}{2} \binom{4}{2} \binom{2}{2} = 15 \cdot 6 \cdot 1 = 90$; and profile $(1, 1, 4)$ -type: $3 \cdot \binom{6}{4} \binom{2}{1} \binom{1}{1}$ — assignments: choose which range element has the 4-fibre (3), then $\binom{6}{4} \cdot \binom{2}{1} \cdot 1 \cdot 2!/2! = 3 \cdot 15 \cdot 2 = 90$. Now $360 + 90 + 90 = 540$. The audit trail itself is the lesson: reconcile totals whenever two roads exist. ■

Time-boxing doctrine

EAMCET: 60–75 seconds/item; formulas must be reflexes; skip any item requiring a fresh derivation and return. JEE Main: 2–3 minutes; one transform, one principle. JEE Advanced: 5–8 minutes; expect chained transforms and integer answers — use integrality as a check. ISI/CMI subjective: 20–30 minutes; the bijection or invariant *is* the answer; write in full sentences; verify small cases in the script itself. Across all: never leave a division unjustified — name the uniform fibre.

Exercises

- (EAMCET drill.) From 6 boys and 4 girls, how many committees of 5 have at most 2 girls? [\[EAMCET\]](#)
- (JEE Main.) How many arrangements of INTERMEDIATE keep the vowels in alphabetical order (not necessarily adjacent)? [\[JEE MAIN\]](#)
- (JEE Advanced.) Find the number of ways to choose two distinct subsets A, B of $[n]$ with $A \cap B = \emptyset$, and separately with $A \subsetneq B$. [\[JEE ADVANCED\]](#)
- (ISI.) Prove: the number of subsets of $[n]$ containing no two consecutive integers is F_{n+2} (Fibonacci), and reconcile with Exercise 6-style gap counting: $\sum_r \binom{n-r+1}{r} = F_{n+2}$. [\[ISI\]](#)
- (CMI.) Show that the number of surjections $[n] \rightarrow [k]$ is divisible by $k!$ and interpret the quotient. [\[CMI\]](#)
- (IAT.) A bag contains tickets numbered 1..30. Count the unordered triples whose numbers sum to an even number. [\[IAT\]](#)

7. (Olympiad.) $2n$ points on a circle; show that the number of ways to join them by n chords, allowing crossings, is $(2n - 1)!!$, and that requiring no crossings cuts this to C_n ; compute the ratio's asymptotics with Stirling. [\[OLYMPIAD/CSIR\]](#)

Part II

Structures and Symmetry

ISI, CMI, IAT, CSIR-NET, NBHM, Olympiad

The Twelffold Way and the Stirling Numbers

ELI5 • Twelve flavours of “put things in boxes”

Every distribution puzzle asks three questions. Are the objects distinct (named toys) or identical (plain chocolates)? Are the boxes distinct (named children) or identical (plain bags)? And is there a rule — none, “no box empty,” or “no box gets two”? Three questions with $2 \times 2 \times 3$ answers give twelve puzzles. Some are easy (each toy picks a child), some are sneaky (identical bags!), but there are only twelve, ever. Learn the grid and no distribution problem can ambush you again.

11.1 Functions, injections, surjections — with symmetry

Formally: count functions $f: N \rightarrow K$ with $|N| = n$ (objects, “balls”) and $|K| = k$ (boxes), where we may quotient by the symmetric group S_N (balls identical), by S_K (boxes identical), or both, and may restrict f to be arbitrary, injective, or surjective. Rota’s *twelffold way*:

f counted up to	any f	injective	surjective
nothing (balls, boxes distinct)	k^n	k^n	$k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$
S_N (balls identical)	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
S_K (boxes identical)	$\sum_{j=1}^k \left\{ \begin{smallmatrix} n \\ j \end{smallmatrix} \right\}$	$[n \leq k]$	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$
$S_N \times S_K$ (both identical)	$p_{\leq k}(n)$	$[n \leq k]$	$p_k(n)$

Here $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the Stirling number of the second kind (defined below), $p_k(n)$ the number of partitions of the integer n into exactly k parts (Chapter 18), and $[\cdot]$ the Iverson bracket. Rows 1–2 are Part I; rows 3–4 are the new mathematics of Part II.

11.2 Stirling numbers of the second kind

Definition 11.1. $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the number of partitions of an n -set into k nonempty unlabeled blocks.

Theorem 11.2 (Basic laws). (a) *Recurrence:* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$, with $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$, $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 0$ ($n \geq 1$).

(b) *Surjection link:* $\text{Sur}(n, k) = k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, hence

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n.$$

- (c) *Basis change*: $x^n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^{\underline{k}}$ as polynomial identity.
 (d) *Bell numbers*: $B_n := \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ (all set partitions) satisfy $B_{n+1} = \sum_{j=0}^n \binom{n}{j} B_j$.

Proof. (a) Element n either joins one of the k blocks of a partition of $[n-1]$ into k blocks ($k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$ ways) or forms a singleton block atop a partition into $k-1$ blocks. (b) A surjection is a partition into k blocks (the fibres) plus a bijection blocks $\rightarrow$ range: the labelling action of S_k on block-labelled partitions is free. Then Theorem 7.2 divides by $k!$. (c) Both sides count functions $[n] \rightarrow X$ for a variable-size codomain: for each positive integer x , functions $[n] \rightarrow [x]$ number x^n ; classify by image size k : choose the image ($\binom{x}{k}$), then a surjection onto it ($k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$); and $\binom{x}{k} k! = x^{\underline{k}}$. Two polynomials in x agreeing at all positive integers agree identically. (d) Classify partitions of $[n+1]$ by the block of the element $n+1$: choose its $n-j$ companions ($\binom{n}{n-j} = \binom{n}{j}$), partition the remaining j elements freely (B_j). $\square$

Statement (c) is the conceptual heart: the Stirling numbers of the second kind are the change-of-basis matrix from falling factorials to monomials in the polynomial ring. The inverse change of basis defines the first kind:

11.3 Stirling numbers of the first kind

Definition 11.3. The (unsigned) Stirling number of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ is the number of permutations of $[n]$ with exactly k cycles.

- Theorem 11.4.** (a) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] + (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right]$.
 (b) $x^{\overline{n}} = x(x+1) \cdots (x+n-1) = \sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$, and $x^{\underline{n}} = \sum_k (-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$.
 (c) The matrices $\left[\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \right]$ and $\left[(-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \right]$ are mutually inverse lower-triangular matrices.

Proof. (a) Element n is either a fixed point (a 1-cycle) over a $(k-1)$ -cycle permutation of $[n-1]$, or is inserted into an existing cycle structure: insertion after any of the $n-1$ existing elements uniquely extends a k -cycle permutation of $[n-1]$. (b) Induction via (a): multiplying $x^{\overline{n-1}}$ by $(x+n-1)$ reproduces exactly the recurrence in the coefficients; the falling version substitutes $x \mapsto -x$ and clears signs. (c) Compose the basis changes of Theorems 11.2(c) and (b): monomials $\rightarrow$ falling factorials $\rightarrow$ monomials is the identity. $\square$

Worked Example 11.1 (*CSIR pattern*). Compute the number of permutations of $[6]$ with exactly 3 cycles, and verify against the cycle-type census.

Solution. Recurrence table or census. Census: cycle types with 3 cycles of total 6: $(4, 1, 1)$: $\frac{6!}{4 \cdot 1 \cdot 1 \cdot 2!} = 90$; $(3, 2, 1)$: $\frac{6!}{3 \cdot 2 \cdot 1} = 120$; $(2, 2, 2)$: $\frac{6!}{2^3 3!} = 15$. Total $\left[\begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right] = 225$. $\blacksquare$

Worked Example 11.2 (*Distribution audit, ISI pattern*). In how many ways can 7 distinct books be tied into 3 identical parcels, none empty? Into at most 3 parcels?

Solution. Exactly 3: $\left\{ \begin{smallmatrix} 7 \\ 3 \end{smallmatrix} \right\}$. From the explicit sum: $\frac{1}{3!}(3^7 - 3 \cdot 2^7 + 3) = \frac{2187-384+3}{6} = \frac{1806}{6} = 301$. At most 3: $\left\{ \begin{smallmatrix} 7 \\ 1 \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} 7 \\ 2 \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} 7 \\ 3 \end{smallmatrix} \right\} = 1 + 63 + 301 = 365$. $\blacksquare$

The Abstract Viewpoint • Two triangles, one umbral mirror

The Stirling pairs exhibit a pervasive duality. Second kind: set *partitions*; first kind: *cycles*; and the exponential generating functions (Chapter 12) are $\sum_n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$ and $\sum_n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{x^n}{n!} = \frac{(\log \frac{1}{1-x})^k}{k!}$: the functions $e^x - 1$ and $\log \frac{1}{1-x}$ are compositional inverses of each

other's negatives — Lagrange inversion in combinatorial dress, and the formal reason the two matrices invert each other. In the calculus of finite differences, (c) of Theorem 11.2 reads Δ^k -coefficients: $\{n\}_k = \frac{\Delta^k 0^n}{k!}$, the k -th difference of the monomial at 0. Categorically, the second-kind triangle is the decategorified bar construction relating the free commutative monoid to the free set; species theory (Chapter 13) states this as $E \circ E_{\geq 1} = \text{partitions}$, where E is the “set” species — substitution of species decategorifies to composition of EGFs, and the Stirling identities become trivialities *about* e^x and $\log$. Finally, in the partition lattice Π_n , the Whitney numbers of the second kind are $\{n\}_k$ and the Möbius function alternates through first-kind values: Chapter 14 closes this loop.

Twelfold triage

Given any distribution problem: write the triple (objects distinct?, boxes distinct?, constraint). Then: DD-any $\rightarrow k^n$; DD-onto $\rightarrow k! \{n\}_k$; DD-injective $\rightarrow k^n$; ID-any $\rightarrow \text{stars-bars}$; ID-onto $\rightarrow \binom{n-1}{k-1}$; DI-any/onto $\rightarrow \text{Stirling sums}$; $\Pi \rightarrow \text{partitions}$ (Chapter 18). The two cells students misfile: DI (distinct objects, identical boxes) — never $k^n/k!$; and “onto” with distinct objects — never stars and bars. Landmark values: $\{n\}_2 = 2^{n-1} - 1$, $\{n\}_{n-1} = \binom{n}{2}$, $[n]_{n-1} = \binom{n}{2}$, $[n]_1 = (n-1)!$, $B_1..B_6 = 1, 2, 5, 15, 52, 203$.

Exercises

1. Prove $\{n\}_2 = 2^{n-1} - 1$ and $\{n\}_{n-1} = \binom{n}{2}$ bijectively. [JEE ADVANCED/ISI]
2. Ten students are split into study groups (any number of groups, none empty, groups unlabeled). How many ways? Express via Bell and compute B_{10} from the recurrence ($B_{10} = 115975$). [ISI]
3. Show $\sum_k \{n\}_k = n!$ in two ways: by the census and by setting $x = 1$ in Theorem 11.4(b). [CSIR]
4. Prove that $\{n+1\}_k = \sum_j \binom{n}{j} \{j\}_{k-1}$ by classifying the block containing $n+1$. [CMI]
5. Expand x^4 in falling factorials and use it to sum $\sum_{x=1}^N x^4$ via hockey stick. [ISI/CMI]
6. Show the number of ways to distribute n distinct objects into k identical boxes with every box holding at least 2 objects satisfies an inclusion–exclusion over singleton-boxes; derive a formula for $k = 2$. [OLYMPIAD]
7. (NBHM.) Prove that $B_p \equiv 2 \pmod{p}$ for prime p (Touchard's congruence base case) using the action of $\mathbb{Z}/p$ on set partitions of $\mathbb{Z}/p$ by rotation and counting fixed partitions. [NBHM/OLYMPIAD]

Generating Functions: Counting Becomes Algebra

ELI5 • A clothesline for numbers

Suppose your counting problem has one answer for each size: a_0 ways for size 0, a_1 for size 1, a_2 for size 2, $\dots$. Instead of carrying an infinite list, hang the whole list on a single clothesline: $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$. The powers of x are just clothespins that keep the numbers in order. The magic: combining two counting problems (choose an apple-bag *and* an orange-bag) becomes *multiplying* their clotheslines, because multiplying power series automatically pairs up sizes that add. One multiplication does infinitely many convolution sums at once.

12.1 Formal power series

Definition 12.1. The ring $\mathbb{Q}[[x]]$ of *formal power series* consists of expressions $A(x) = \sum_{n \geq 0} a_n x^n$ with $a_n \in \mathbb{Q}$, added coefficientwise and multiplied by the Cauchy convolution $(\sum a_n x^n)(\sum b_n x^n) = \sum_n (\sum_{k=0}^n a_k b_{n-k}) x^n$. Write $[x^n]A(x) = a_n$ for coefficient extraction. No convergence is asserted or needed: x is a bookkeeping symbol, and all identities are identities of coefficient sequences.

Proposition 12.2. $A(x)$ is invertible in $\mathbb{Q}[[x]]$ iff $a_0 \neq 0$; in particular $\frac{1}{1-x} = \sum_{n \geq 0} x^n$. Composition $A(B(x))$ is defined whenever $b_0 = 0$. The formal derivative obeys the usual rules. The generalized binomial series holds formally for any $\alpha \in \mathbb{Q}$:

$$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha^n}{n!},$$

and in particular the negative binomial law

$$\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^n.$$

Proof. Invertibility: solve $\sum_k a_k b_{n-k} = [n=0]$ recursively for b_n ; $b_0 = 1/a_0$ is the only obstruction. The negative binomial law has a counting proof that we prefer to the formal one: $\frac{1}{(1-x)^k} = (\sum_{m \geq 0} x^m)^k$, and $[x^n]$ of the k -fold product counts the ways to write $n = m_1 + \dots + m_k$ with $m_i \geq 0$: stars and bars, $\binom{n+k-1}{k-1}$. Thus Theorem 6.1 is the expansion of $(1-x)^{-k}$ — the first exhibit of counting-as-algebra. $\square$

12.2 Ordinary generating functions and the product principle

The OGF dictionary

Model each “component” of a structure by a series whose x -exponent tracks size. Then, for building a structure by choosing one option from each of several *independent* components with sizes adding: *alternatives add series; independent stages multiply series; one-or-more sequences of a component with no size-0 piece give $\frac{1}{1-A(x)}$* . Constraints on multiplicities become choices of which powers appear: “even number of red balls” $\rightarrow 1+x^2+x^4+\cdots = \frac{1}{1-x^2}$; “between 2 and 5 blue” $\rightarrow x^2+x^3+x^4+x^5$; “any number” $\rightarrow \frac{1}{1-x}$; “0 or 1 (distinct-type choice)” $\rightarrow 1+x$.

Worked Example 12.1 (*Bounded distributions redone*). Coefficient method for Chapter 7’s bounded sum: solutions of $x_1 + \cdots + x_4 = 30$, $0 \leq x_i \leq 10$, number

$$[x^{30}] \left(\frac{1-x^{11}}{1-x} \right)^4 = [x^{30}] (1-x^{11})^4 \sum_m \binom{m+3}{3} x^m = \binom{33}{3} - 4 \binom{22}{3} + 6 \binom{11}{3} = 286.$$

Solution. Expanding $(1-x^{11})^4 = \sum_j (-1)^j \binom{4}{j} x^{11j}$ reproduces, term by term, the inclusion–exclusion of Chapter 7: the algebra *is* PIE. One formula, no cases, and upper bounds cost nothing. ■

Worked Example 12.2 (*A selection with parity constraints, ISI pattern*). How many ways to select n fruits from unlimited apples, bananas, oranges, pears if apples must be even in number, bananas a multiple of 5, oranges at most 4, and pears at most 1?

Solution.

$$A(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot (1+x) = \frac{1+x}{(1-x^2)(1-x)} = \frac{1}{(1-x)^2},$$

so the answer is $[x^n](1-x)^{-2} = n+1$ — a collapse no case analysis would reveal in advance. (This is Wilf’s beloved example, and a standard ISI interview set piece.) ■

12.3 Solving recurrences

Worked Example 12.3 (*Fibonacci in closed form*). Solve $a_n = a_{n-1} + a_{n-2}$, $a_1 = 2$, $a_2 = 3$ (Chapter 2’s no-consecutive-ones count).

Solution. Let $A(x) = \sum_{n \geq 1} a_n x^n$. Multiply the recurrence by x^n , sum over $n \geq 3$: $A - 2x - 3x^2 = x(A - 2x) + x^2 A$, so $A(x) = \frac{2x + x^2}{1 - x - x^2}$. Partial fractions over the roots of $1 - x - x^2$ — equivalently, since $a_n = F_{n+2}$ (Fibonacci with $F_1 = F_2 = 1$) — give Binet’s form

$$a_n = \frac{\varphi^{n+2} - \psi^{n+2}}{\sqrt{5}}, \quad \varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}.$$

The template: linear recurrence with constant coefficients $\iff$ rational generating function with denominator the reversed characteristic polynomial $\iff$ closed form as a sum of $p_i(n)\lambda_i^n$ over characteristic roots λ_i with multiplicities. All three descriptions are interconvertible, and CSIR objective papers test exactly this dictionary. ■

Worked Example 12.4 (*Catalan, nonlinearly*). From the first-return decomposition $C_{n+1} = \sum_k C_k C_{n-k}$: the OGF $C(x) = \sum C_n x^n$ satisfies $C = 1 + xC^2$, whence $C(x) = \frac{1 - \sqrt{1-4x}}{2x}$ (choosing the root with $C(0) = 1$), and the generalized binomial series yields $C_n = \frac{1}{n+1} \binom{2n}{n}$ again — this time with no reflection, by pure series algebra.

Solution. Expand $(1-4x)^{1/2} = \sum_n \binom{1/2}{n} (-4)^n x^n$ and simplify $\binom{1/2}{n} (-4)^n = -\frac{2}{2n-1} \binom{2n}{n} \cdot \frac{1}{2}$ (routine falling-factorial algebra); the constant and linear normalizations produce exactly C_n . The exercise of carrying this computation through once, in full, is compulsory for ISI/CMI aspirants. ■

12.4 Exponential generating functions

For *labelled* structures — where the atoms are distinguishable and merging two structures requires distributing labels — the right clothesline is exponential.

Definition 12.3. The EGF of (a_n) is $\hat{A}(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

Theorem 12.4 (EGF product = labelled merge). If a_n counts A -structures and b_n counts B -structures on labelled n -sets, then $\hat{A}\hat{B}$ is the EGF of structures “split the label set into two blocks, put an A -structure on the first and a B -structure on the second”: $[x^n/n!] \hat{A}\hat{B} = \sum_k \binom{n}{k} a_k b_{n-k}$.

Proof. Multiply and compare: $\frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} = \frac{1}{n!} \binom{n}{k} a_k b_{n-k}$; the binomial factor is exactly the choice of which labels go left. □

Immediate payoffs. Functions: each label independently chooses among k images; the one-atom EGF is e^x per box, and $(e^x)^k = e^{kx}$ recovers k^n . Surjections: each box must be nonempty: $(e^x - 1)^k$, so $\text{Sur}(n, k) = n! [x^n] (e^x - 1)^k$, and expanding by the binomial theorem re-derives Theorem 7.3 in one line. Derangements: a permutation is a set of cycles; excluding 1-cycles (below) gives $\hat{D}(x) = \frac{e^{-x}}{1-x}$, whence $D_n = n! \sum_{j \leq n} \frac{(-1)^j}{j!}$ by the product of the two series — Theorem 7.5 with no inclusion–exclusion at all.

Theorem 12.5 (The exponential formula). Let c_n ($n \geq 1$) count connected structures on a labelled n -set, with EGF $\hat{C}$. Then the EGF of structures that are sets of connected components is $e^{\hat{C}(x)}$.

Proof. $\frac{\hat{C}^k}{k!}$ is the EGF for structures with exactly k components: the k -fold labelled merge of Theorem 12.4 builds an *ordered* k -tuple of components, and division by $k!$ is licensed because components, having nonempty disjoint label sets, are distinguishable — the reordering action is free. Sum over k . □

Applications cascade. Permutations are sets of cycles; cycles on $m \geq 1$ labels number $(m-1)!$, with EGF $\sum_{m \geq 1} \frac{x^m}{m} = \log \frac{1}{1-x}$; so permutations have EGF $e^{\log \frac{1}{1-x}} = \frac{1}{1-x}$: consistency, $n!$ per level. Restrict cycle lengths and read off counts: involutions ($e^{x+x^2/2}$), derangements ($e^{-x}/(1-x)$ via removing the x term: $e^{\log \frac{1}{1-x} - x}$), permutations with all cycles of length > 2 , etc. Set partitions are sets of nonempty blocks ($e^x - 1$ per block): $\sum_n B_n \frac{x^n}{n!} = e^{e^x - 1}$, and the k -block slice $\frac{(e^x - 1)^k}{k!}$ is the Stirling EGF promised in Chapter 11.

The Abstract Viewpoint • What a generating function is

Three ascending answers. (1) A sequence in one variable: bookkeeping. (2) An element of the ring $\mathbb{Q}[[x]]$, where combinatorial constructions (disjoint union, product, sequence,

set, cycle) become algebraic operations ($+$, $\times$, $\frac{1}{1-\cdot}$, $\exp$, $\log \frac{1}{1-\cdot}$): the theory of *species* (Chapter 13) makes this a functorial dictionary, with OGFs decategorifying unlabelled structures and EGFs labelled ones. (3) An analytic function near 0, when the series converges: then singularity analysis reads asymptotics from the geometry of the nearest singularity — $\frac{1}{1-x-x^2}$ has its singularity at $1/\varphi$, forcing $a_n \sim c\varphi^n$; $\sqrt{1-4x}$ has a branch point at $1/4$, forcing $C_n \sim \frac{4^n}{\sqrt{\pi n^{3/2}}}$ (the universal $n^{-3/2}$ of tree-like classes). Flajolet–Sedgewick’s *Analytic Combinatorics* develops (3) into a machine; CSIR and NBHM sample its first chapter, and research-level enumeration lives there. The three levels never conflict: formal identities proved at level (2) are inherited at level (3) wherever convergence holds.

Which GF when

Unlabelled/identical atoms, sizes add $\rightarrow$ OGF. Labelled atoms, structures merged by splitting label sets $\rightarrow$ EGF. Words with letter-multiplicity constraints $\rightarrow$ EGF (positions are labels!): the number of n -letter words over $\{a, b, c\}$ with an even number of a ’s is $n! [x^n] \frac{e^x + e^{-x}}{2} e^{2x} = \frac{3^n + 1}{2}$ — a two-line solution to a JEE-Advanced-hard problem. Recurrences $\rightarrow$ OGF, rational algebra. “Sets of components” $\rightarrow$ exponential formula. Bounded multiplicities $\rightarrow$ polynomial factors; congruence multiplicities $\rightarrow$ roots-of-unity sections $\frac{1}{m} \sum_j A(\omega^j x)$.

Exercises

1. Compute $[x^{20}] \frac{1}{(1-x)(1-x^2)(1-x^5)}$: the number of ways to pay 20 rupees with 1, 2, 5 coins. (Organize as a small table; this is the money-changing OGF.) [JEE ADVANCED/ISI]
2. Prove via EGFs: the number of n -words over $\{a, b, c, d\}$ with an odd number of a ’s and an even number of b ’s is $\frac{4^n - (-2)^n \cdot 0 + 2^n \cdot ?}{4}$ — derive the exact closed form $\frac{4^n - 0 - 0 + \dots}{4}$ yourself and check $n = 1, 2$. [ISI/CMI]
3. Solve $a_n = 3a_{n-1} - 2a_{n-2}$, $a_0 = 2, a_1 = 3$, by OGF and verify with the characteristic-root method. [CSIR]
4. Show that the EGF of involutions is $e^{x+x^2/2}$ and re-derive the recurrence $I(n) = I(n-1) + (n-1)I(n-2)$ by differentiating. [CSIR]
5. Use the roots-of-unity filter on $(1+x)^n$ to evaluate $\sum_{k \equiv 0(3)} \binom{n}{k}$. [JEE ADVANCED/OLYMPIAD]
6. Carry out in full the expansion of $\frac{1-\sqrt{1-4x}}{2x}$ to obtain $C_n = \frac{1}{n+1} \binom{2n}{n}$. [ISI/CMI]
7. Derive the EGF identity e^{e^x-1} for Bell numbers and, by differentiating, re-prove $B_{n+1} = \sum_j \binom{n}{j} B_j$. [CSIR/NBHM]
8. (Snake oil practice.) Evaluate $\sum_k \binom{n+k}{2k} 2^{n-k}$ by generating functions in n . [OLYMPIAD]

Recurrences, Transfer Matrices, and a Glimpse of Species

ELI5 • Climbing stairs by remembering one step back

You climb a staircase taking 1 or 2 steps at a time. To count routes to step 10, notice: your last move came from step 9 or step 8. So $(\text{routes to } 10) = (\text{routes to } 9) + (\text{routes to } 8)$. You never need to see the whole staircase — just the two steps behind you. That is a recurrence: a rule for the next answer in terms of a few previous ones. Big counting problems become tiny local rules.

13.1 Designing recurrences

A recurrence is designed, not found. The design step is the choice of *state*: what minimal information about a partial structure determines how it can be extended?

Worked Example 13.1 (*State design, ISI pattern*). Count ternary strings of length n over $\{0, 1, 2\}$ with no “12” as a consecutive substring.

Solution. State: the last symbol (enough, since the forbidden pattern has length 2). Let u_n, v_n, w_n count valid strings ending in 0, 1, 2. Transitions: append 0: always allowed; append 1: always; append 2: forbidden only after 1. Thus

$$u_n = u_{n-1} + v_{n-1} + w_{n-1}, \quad v_n = u_{n-1} + v_{n-1} + w_{n-1}, \quad w_n = u_{n-1} + w_{n-1}.$$

With $t_n = u_n + v_n + w_n$: $t_n = 3t_{n-1} - v_{n-1} = 3t_{n-1} - t_{n-2}$ (since $v_{n-1} = t_{n-2}$). Values: $t_1 = 3$, $t_2 = 8$, then 21, 55, 144, ... — every second Fibonacci-like; in fact $t_n = F_{2n+2}$ with suitable indexing, a pleasant surprise the recurrence surfaces automatically. ■

13.2 The transfer-matrix method

The example generalizes: when validity of a string is checked by a finite-memory scanner, counting is linear algebra.

Theorem 13.1 (Transfer matrix). *Let a finite state set Q , an initial vector $\mathbf{s} \in \mathbb{Q}^Q$, a final vector $\mathbf{f}$, and a matrix T with $T_{qq'}$ = number of admissible letters that move state q to q' be given, so that admissible strings of length n correspond to walks of length n in the state digraph. Then the number of admissible strings of length n is $\mathbf{s}^\top T^n \mathbf{f}$, and the generating function $\sum_n \mathbf{s}^\top T^n \mathbf{f} x^n = \mathbf{s}^\top (I - xT)^{-1} \mathbf{f}$ is rational with denominator dividing $\det(I - xT)$.*

Proof. $(T^n)_{qq'}$ counts length- n walks from q to q' : induction on n via matrix multiplication, which is exactly the sum-over-intermediate-state decomposition (sum and product rules in matrix clothing). The series identity is the geometric series $\sum_n (xT)^n = (I - xT)^{-1}$, valid formally; Cramer’s rule exhibits the rational form. □

Consequences worth owning: every finite-automaton-definable class of words has a rational OGF, hence a linear recurrence with constant coefficients, hence $\sum p_i(n)\lambda_i^n$ asymptotics with λ_i the eigenvalues of T — the dominant eigenvalue (Perron–Frobenius, for nonnegative T) governs growth. This is the conceptual backbone behind “no two adjacent” counts, tiling counts, and the paving of CSIR-style dynamical/Markov questions.

Worked Example 13.2 (*Tilings*). Count tilings of a $2 \times n$ strip by dominoes; then by dominoes and 2×2 squares.

Solution. Dominoes: scan column by column; a column boundary is either flush (state A) or has one protruding half of a horizontal pair — but for $2 \times n$ the clean recurrence needs only flush states: last column is one vertical domino (f_{n-1}) or two horizontal dominoes spanning two columns (f_{n-2}): $f_n = f_{n-1} + f_{n-2}$, $f_0 = f_1 = 1$: Fibonacci. Allowing also the 2×2 square adds a second two-column option: $g_n = g_{n-1} + 2g_{n-2}$, $g_0 = g_1 = 1$, so $g_n = \frac{2^{n+1} + (-1)^n}{3}$ by characteristic roots 2, -1 . ■

13.3 Nonlinear and full-history recurrences

Not all recurrences are linear. The Catalan recurrence $C_{n+1} = \sum C_k C_{n-k}$ is quadratic (convolutional), signalling a *multiplicative* equation for the OGF, $C = 1 + xC^2$. Bell’s $B_{n+1} = \sum_j \binom{n}{j} B_j$ is a binomial (EGF-multiplicative) full-history recurrence, signalling $\hat{B}' = e^x \hat{B}$. The taxonomy:

Recurrence shape	GF equation
constant-coefficient linear	rational OGF
convolution $\sum a_k a_{n-k}$	polynomial equation in $A(x)$ (algebraic)
binomial convolution	product/ODE for EGF
n -dependent linear coefficients	linear ODE for OGF/EGF (holonomic)

The last row (holonomic/P-recursive sequences: e.g. $D_n = nD_{n-1} + (-1)^n$, or $\binom{2n}{n}$ with $(n+1)a_{n+1} = 2(2n+1)a_n$) covers most sequences met in practice; Zeilberger’s algorithm mechanizes identities among them — a name CSIR aspirants should recognize.

13.4 A structured glimpse: combinatorial species

We can now honour the promise of Chapters 1 and 12: a formalism in which “structure” is itself a mathematical object.

Definition 13.2 (Joyal). A *species* F assigns to each finite set U a finite set $F[U]$ of “ F -structures on U ,” and to each bijection $\sigma: U \rightarrow V$ a bijection $F[\sigma]: F[U] \rightarrow F[V]$, functorially ($F[\text{id}] = \text{id}$, $F[\sigma\tau] = F[\sigma]F[\tau]$). Its EGF is $F(x) = \sum_n |F[[n]]| \frac{x^n}{n!}$.

Examples: E (sets: exactly one structure on any U), $E(x) = e^x$; X (singletons), $X(x) = x$; L (linear orders), $L(x) = \frac{1}{1-x}$; C (cyclic orders), $C(x) = \log \frac{1}{1-x}$; S (permutations), $S(x) = \frac{1}{1-x}$; Par (set partitions), $\text{Par}(x) = e^{e^x - 1}$.

Theorem 13.3 (Operations). *Sum, product, and composition of species satisfy $(F + G)(x) = F(x) + G(x)$, $(F \cdot G)(x) = F(x)G(x)$ (product structure: split U into two blocks, an F -structure*

on one, G on the other), and, when $G[\emptyset] = \emptyset$, $(F \circ G)(x) = F(G(x))$ (composite structure: partition U , a G -structure on each block, an F -structure on the set of blocks).

Proof sketch. Sum and product are Theorem 12.4. Composition: an $(F \circ G)$ -structure on $[n]$ is a partition into k blocks with G -structures and an F -structure on the k -set of blocks; summing over block-size compositions and dividing by the free reordering reproduces the coefficients of $F(G(x))$ exactly as in the exponential formula, of which this theorem is the vast generalization ($E \circ G$ recovers Theorem 12.5). $\square$

Now cherished identities become structural one-liners: $S = E \circ C$ (a permutation is a set of cycles): $\frac{1}{1-x} = e^{\log \frac{1}{1-x}}$; $\text{Par} = E \circ E_{\geq 1}$: e^{e^x-1} ; derangements $= E \circ C_{\geq 2}$: $\frac{e^{-x}}{1-x}$; labelled rooted trees satisfy $A = X \cdot (E \circ A)$, giving $A(x) = xe^{A(x)}$, and Lagrange inversion extracts Cayley’s formula: the number of labelled rooted trees on n vertices is n^{n-1} (hence n^{n-2} unrooted — Cayley’s theorem, an NBHM favourite whose bijective Prüfer-code proof the reader should also learn).

The Abstract Viewpoint • Why species matter

Species are the categorification of EGFs: identities of series lift to natural isomorphisms of functors, and a “bijective proof” is precisely a natural isomorphism. The formalism also sees what EGFs cannot: two species with equal EGFs need not be isomorphic (L vs. S : linear orders and permutations both give $\frac{1}{1-x}$, yet as functors they differ — naturality detects that a linear order has no canonical automorphisms while permutations do). The finer invariant, the *cycle index series* Z_F , remembers the action of relabelling symmetries; specializing Z_F yields both the EGF (labelled count) and the OGF of unlabelled structures — and for $F = \text{colourings}$ acted on by a group, Z becomes Pólya’s cycle index (Chapter 16). Thus species unify Chapters 12 and 16 into one functorial picture; Bergeron–Labelle–Leroux is the standard reference, and the viewpoint quietly organizes modern algebraic combinatorics from symmetric functions to operads.

Recurrence craft for examinations

(1) Choose the state to be the minimal suffix information (last letter; last column profile; current remainder mod m). (2) Check the recurrence on brute-forced small cases before trusting it. (3) For closed forms: linear constant-coefficient $\rightarrow$ characteristic roots; convolution $\rightarrow$ OGF functional equation; binomial convolution $\rightarrow$ EGF ODE. (4) Growth-rate questions: dominant eigenvalue/root only — do not compute the full closed form when only $\lim a_{n+1}/a_n$ is asked (a common CSIR time sink). (5) Cayley’s n^{n-2} , derangement recurrences, Fibonacci tilings, and the staircase are permanent residents of ISI/CMI/CSIR papers.

Exercises

- Count binary strings of length n avoiding “11” and report the transfer matrix, its characteristic polynomial, and the growth constant. [ISI/CSIR]
- Tilings of $1 \times n$ by tiles of length 1, 2, 3: recurrence, OGF, and the growth rate to two decimals. [ISI]
- Design a recurrence for the number of n -digit strings over $\{0, \dots, 9\}$ whose digit sum is divisible by 3, and solve it via roots-of-unity instead. [JEE ADVANCED/ISI]
- Show that the number of compositions of n into parts ≥ 2 is F_{n-1} , by recurrence and by OGF $\frac{x^2/(1-x)}{1-x^2/(1-x)}$ simplification. [ISI]

5. Prove Cayley's formula via the Prüfer bijection: labelled trees on $[n] \leftrightarrow$ sequences in $[n]^{n-2}$. [\[NBHM/OLYMPIAD\]](#)
6. From $A(x) = xe^{A(x)}$, apply Lagrange inversion $[x^n]A = \frac{1}{n}[t^{n-1}]e^{nt}$ to get $n^{n-1}/n! \cdot n!$; write the argument carefully. [\[CSIR/NBHM\]](#)
7. (Olympiad.) A $3 \times n$ board is tiled by dominoes. Set up the transfer matrix on column-interface states and show $t_n = 4t_{n-2} - t_{n-4}$ for even n . [\[OLYMPIAD\]](#)

Möbius Inversion: Inclusion–Exclusion Grown Up

ELI5 • Un-mixing cumulative totals

A shop reports cumulative sales up to each day, but you want daily sales. Easy: subtract yesterday's total from today's. Now imagine the “days” are not in a line but in a family tree — each node's report includes all its descendants. To recover each node's own figure, you must subtract children, add back grandchildren shared twice, and so on: a pattern of + and – determined by the *shape* of the tree. Möbius inversion is the universal recipe for the right pattern of signs on any shape of hierarchy.

14.1 Posets, incidence algebras, and μ

Definition 14.1. A *partially ordered set* (poset) is a set P with a reflexive, antisymmetric, transitive relation $\leq$. We assume P locally finite: every interval $[x, y] = \{z : x \leq z \leq y\}$ is finite. The *incidence algebra* $I(P)$ consists of functions $f(x, y)$ defined for $x \leq y$, with convolution $(f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y)$. Its identity is $\delta(x, y) = [x = y]$; the *zeta function* is $\zeta(x, y) = 1$ for all $x \leq y$.

Theorem 14.2 (Existence of μ ; Möbius inversion). ζ is invertible in $I(P)$; its inverse μ is computed recursively by

$$\mu(x, x) = 1, \quad \mu(x, y) = - \sum_{x \leq z < y} \mu(x, z) \quad (x < y).$$

Consequently, for functions $f, g: P \rightarrow \mathbb{Q}$ (with appropriate finiteness):

$$g(y) = \sum_{x \leq y} f(x) \quad \forall y \quad \Longleftrightarrow \quad f(y) = \sum_{x \leq y} g(x) \mu(x, y) \quad \forall y.$$

Proof. The recursion defines a left inverse: $(\mu * \zeta)(x, y) = \sum_{x \leq z \leq y} \mu(x, z) = [x = y]$ by construction; in an incidence algebra a one-sided inverse of ζ is two-sided (triangularity: induct on interval length). The inversion formula is the associativity computation $f = f * \delta = f * (\zeta * \mu) = (f * \zeta) * \mu = g * \mu$ written pointwise, treating f, g as row vectors. $\square$

14.2 The three canonical posets

14.2.1 Chains: telescoping

$P = \mathbb{N}$ with the usual order: $\mu(m, n) = 1$ if $n = m$, -1 if $n = m + 1$, else 0. Inversion says: cumulative sums are undone by successive differences — the shop of the ELI5 box, and the fundamental theorem of the difference calculus.

14.2.2 The Boolean lattice: inclusion–exclusion

$P = 2^{[k]}$ ordered by inclusion: intervals $[S, T]$ are Boolean lattices of rank $|T \setminus S|$, and

Proposition 14.3. $\mu(S, T) = (-1)^{|T \setminus S|}$.

Proof. Verify the recursion: $\sum_{S \subseteq Z \subseteq T} (-1)^{|Z \setminus S|} = \sum_j \binom{|T \setminus S|}{j} (-1)^j = [S = T]$, the alternating row sum. $\square$

Möbius inversion over $2^{[k]}$ is verbatim the generalized inclusion–exclusion of Chapter 7 (the “exactly m properties” formula included); PIE is thus one column of a universal theory, as promised.

14.2.3 The divisor lattice: number theory

$P = \mathbb{N}_{\geq 1}$ ordered by divisibility. Intervals $[d, n]$ are isomorphic to divisor lattices of n/d , and since n/d factors as a product of prime powers, the interval is a product of chains — hence, by the product theorem below, $\mu(d, n) = \mu(n/d)$ where μ is the classical Möbius function

$$\mu(m) = \begin{cases} (-1)^r & m \text{ squarefree with } r \text{ prime factors,} \\ 0 & m \text{ not squarefree.} \end{cases}$$

Theorem 14.4 (Product theorem). $\mu_{P \times Q}((x, y), (x', y')) = \mu_P(x, x') \mu_Q(y, y')$.

Proof. The incidence algebra of a product contains the tensor product; the tensor of the two inverses of ζ inverts $\zeta_{P \times Q}$ since $\zeta_{P \times Q} = \zeta_P \otimes \zeta_Q$: check by convolving and factoring the double sum. $\square$

Classical Möbius inversion

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(n/d) g(d)$$

now falls out, and with it the standard harvest:

Worked Example 14.1 (*Totient, again*). From $n = \sum_{d|n} \varphi(d)$ (classify $[n]$ by gcd with n : exactly $\varphi(n/d)$ elements have gcd d ; re-index), inversion gives $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$ — Theorem 7.7 re-proved structurally.

Worked Example 14.2 (*Counting primitive objects: necklaces and polynomials*). (i) The number of *aperiodic* words of length n over k letters (words equal to none of their nontrivial rotations) is $\sum_{d|n} \mu(n/d) k^d$; dividing by n counts aperiodic necklaces (Lyndon words): $L_k(n) = \frac{1}{n} \sum_{d|n} \mu(n/d) k^d$. (ii) The same expression counts monic irreducible polynomials of degree n over $\mathbb{F}_k$ (k a prime power).

Solution. (i) Every word of length n is a power $w = u^{n/d}$ of a unique aperiodic word u of some length $d \mid n$; hence $k^n = \sum_{d|n} a(d)$ where $a(d)$ counts aperiodic words of length d . Invert. Rotations act freely on aperiodic words, so necklaces $= a(n)/n$. (ii) In $\mathbb{F}_k[x]$, $x^{k^n} - x$ is the product of all monic irreducibles of degree dividing n (a standard field-theory fact); comparing degrees, $k^n = \sum_{d|n} d N_d$ with N_d the irreducible count. Invert: $n N_n = \sum_{d|n} \mu(n/d) k^d$. The coincidence with (i) is no accident — both count orbits of Frobenius-type cyclic actions — and it is a staple CSIR/NBHM question in both wardrobes. $\blacksquare$

14.3 The partition lattice

The poset Π_n of set partitions of $[n]$, ordered by refinement ($\pi \leq \sigma$ iff every block of π is contained in a block of σ), is the home lattice of Chapter 11.

Theorem 14.5. *In Π_n , with $\hat{0}$ the all-singletons partition and $\hat{1}$ the one-block partition, $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$. More generally, if σ has blocks of sizes $b_1, \dots, b_k$, then $\mu(\hat{0}, \sigma) = \prod_{i=1}^k (-1)^{b_i-1}(b_i-1)!$.*

Proof sketch. The interval $[\hat{0}, \sigma]$ factors as $\prod_i \Pi_{b_i}$ (refinements act independently within blocks); by the product theorem it suffices to compute $\mu(\hat{0}, \hat{1})$ in Π_n . Use the EGF/exponential-formula route: the defining recursion for μ mirrors the relation between “all structures” and “connected structures,” and taking $F = E_{\geq 1}$ with $e^x - 1 = \text{block-EGF}$, one finds the Möbius values are the coefficients of the compositional inverse $\log(1+x)$: namely $(-1)^{n-1}(n-1)!/n!$ scaled — i.e. $\mu(\hat{0}, \hat{1}) = n! [x^n] \log(1+x) = (-1)^{n-1}(n-1)!$. Filling in this sketch rigorously (via Theorem 14.2’s recursion and induction) is Exercise 6, one of the best structured exercises in this book. $\square$

The Abstract Viewpoint • Rota’s programme and beyond

Rota (1964) recast enumeration as linear algebra over posets: counting formulas are statements about ζ and μ in incidence algebras, and “sign patterns” are geometric invariants of order structures. Highlights of the landscape the reader now borders: (1) *Topology*: by P. Hall’s theorem, $\mu(x, y)$ is the reduced Euler characteristic of the order complex of the open interval (x, y) — signs literally come from topology, and the partition lattice’s $(n-1)!$ reflects the homology of braid arrangements. (2) *Cumulants*: over Π_n , inversion converts moments into cumulants; over the non-crossing lattice $NC(n)$ (Möbius values: signed Catalans) it yields *free* cumulants — Voiculescu’s free probability is Chapter 9’s non-crossing world made analytic. (3) *Arithmetic*: the divisor-lattice μ drives Dirichlet series via $\frac{1}{\zeta(s)} = \sum \mu(n)n^{-s}$; incidence algebras are the abstract Dirichlet rings. One theorem, three sciences.

Recognition

“Cumulative over a hierarchy, recover pointwise” $\rightarrow$ Möbius inversion. “Aperiodic/primitive objects” $\rightarrow$ divisor-lattice inversion with μ . “Irreducible polynomial counts over $\mathbb{F}_q$ ” $\rightarrow$ same formula — CSIR asks this most years in some guise. Boolean-lattice problems are just PIE; do not deploy heavy machinery where Chapter 7 suffices. Memorize: $\sum_{d|n} \mu(d) = [n=1]$; $\varphi = \mu * \text{id}$; $L_2(n)$ for small n : 2, 1, 2, 3, 6, 9, 18, 30 ($n = 1..8$).

Exercises

1. Verify $\sum_{d|12} \mu(d) = 0$ and compute $\mu(d, 60)$ for all $d \mid 60$ as classical $\mu(60/d)$. [CSIR]
2. Prove $\sum_{d|n} \mu(d) = [n=1]$ from the recursion, and derive from it the classical inversion formula directly (without incidence algebras). [ISI/CSIR]
3. Count binary Lyndon words of length 6 and list them. [CMI]
4. How many monic irreducible cubics exist over $\mathbb{F}_5$? Verify by counting reducible ones. [CSIR/NBHM]
5. Show that the number of necklaces (rotation only) of n beads in k colours is $\frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d}$ by summing the Lyndon counts $L_k(d)$ over $d \mid n$ and Möbius-massaging — reconciling Chapter 4’s preview with this chapter. [OLYMPIAD/CSIR]

6. Prove Theorem 14.5 for $n = 3, 4$ by hand from the recursion, then in general by induction using the identity $\sum_{\pi \in \Pi_n} \prod_{B \in \pi} (-1)^{|B|-1} (|B| - 1)! = [n = 1]$, which you should recognize as $n! [x^n] e^{\log(1+x)}$. [\[NBHM\]](#)
7. (Classical.) Using inversion over divisors, show that the number of irreducible fractions a/b with $1 \leq a \leq b \leq n$ is $\sum_{b \leq n} \varphi(b)$, and estimate its growth $\sim 3n^2/\pi^2$ (state, don't prove, $\sum \mu(d)/d^2 = 6/\pi^2$). [\[CSIR\]](#)

Pigeonhole and Double Counting: Counting as Proof

ELI5 • Ten pigeons, nine holes

If ten pigeons fly into nine holes, some hole holds two pigeons. Obvious? Completely. Useless? The opposite: it is one of the sharpest proof weapons in mathematics. The skill is never the principle — it is inventing the *holes*. Choose the right nine boxes and the crowded box tells you a secret about the pigeons.

15.1 The pigeonhole principle and its quantitative form

Theorem 15.1 (Pigeonhole). *If n objects are placed in k boxes, some box contains at least $\lceil n/k \rceil$ objects, and some box contains at most $\lfloor n/k \rfloor$.*

Proof. If every box held $\leq \lceil n/k \rceil - 1$ objects, the total would be at most $k(\lceil n/k \rceil - 1) < k \cdot \frac{n}{k} \cdot \frac{k(\lceil n/k \rceil - 1) + k}{n} \dots$ cleanly: $k(\lceil n/k \rceil - 1) \leq k \cdot \frac{n+k-1}{k} - k = n - 1 < n$. Dually for the minimum. Note this is Proposition 1.3(a) sharpened: an “averaging” statement — some value is at least the mean. $\square$

The averaging reformulation is the mature form: *a set of numbers contains one at least as large as its average, and one at most as large*. Most olympiad pigeonhole is averaging over a cleverly chosen statistic.

Worked Example 15.1 (*Designed boxes, ISI classic*). Show that among any $n + 1$ integers, two differ by a multiple of n ; and that some nonempty consecutive block of any sequence $a_1, \dots, a_n$ has sum divisible by n .

Solution. First: boxes = residues mod n ; $n + 1$ pigeons, n boxes; two share a residue and their difference is divisible by n . Second: consider the $n + 1$ prefix sums $0, s_1, s_1 + s_2$ pattern — precisely $S_0 = 0, S_j = a_1 + \dots + a_j$ for $j = 0..n$: $n + 1$ values, n residue boxes; two prefix sums agree mod n , and the block strictly between them has sum $\equiv 0$. Boxes are residues; pigeons are *prefix sums* — the design step. $\blacksquare$

Theorem 15.2 (Erdős–Szekeres). *Any sequence of $mn + 1$ distinct real numbers contains an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.*

Proof. Label each position i with the pair (u_i, d_i) : the longest increasing, respectively decreasing, subsequence *ending* at i . The labelling is injective: if $i < j$ then $a_i < a_j$ forces $u_j \geq u_i + 1$ and $a_i > a_j$ forces $d_j \geq d_i + 1$; either way $(u_i, d_i) \neq (u_j, d_j)$. If all $u_i \leq m$ and $d_i \leq n$, only mn labels exist for $mn + 1$ positions — contradiction. The bound is sharp: concatenate n decreasing blocks of length m arranged increasingly by block. $\square$

The Erdős–Szekeres proof is pigeonhole at its most elegant: the boxes are label-pairs *manufactured from the problem’s own structure*. The theorem is also the front porch of Ramsey theory: “total disorder is impossible.” A single Ramsey specimen, examinable at olympiad/NBHM level:

Proposition 15.3 ($R(3, 3) \leq 6$). *Among any 6 people, some 3 are mutual acquaintances or some 3 are mutual strangers.*

Proof. Fix person v : among the other 5, pigeonhole gives 3 in one relation to v (say all acquaintances). If any two of those 3 are acquainted, they with v form an acquainted triangle; otherwise the 3 are mutual strangers. Sharp: the pentagon’s acquaintance graph C_5 on 5 people has neither. $\square$

15.2 Double counting

To *double count* is to evaluate one quantity two ways and equate. We have used it silently (Legendre’s formula; committee–chair); we now make it a named method with its standard incarnation:

Theorem 15.4 (Handshake lemma; bipartite counting). (i) *In any finite graph, $\sum_v \deg(v) = 2|E|$; hence the number of odd-degree vertices is even.* (ii) *For finite sets A, B and a relation $R \subseteq A \times B$, $\sum_{a \in A} |R(a)| = |R| = \sum_{b \in B} |R^{-1}(b)|$: count incident pairs from either side.*

Proof. Both are the sum rule applied to the edge/incidence set partitioned two ways. $\square$

Worked Example 15.2 (*A design-theoretic count, CSIR/NBHM pattern*). In a club of v members, committees (blocks) of size k are formed so that every pair of members serves together in exactly λ committees. Show that every member serves in the same number $r = \lambda \frac{v-1}{k-1}$ of committees, and the total number of committees is $b = \lambda \frac{v(v-1)}{k(k-1)}$; deduce Fisher-type divisibility conditions.

Solution. Fix member x ; double count pairs (committee $C \ni x$, member $y \in C \setminus x$): committee-first gives $r_x(k-1)$; pair-first gives $\lambda(v-1)$. So $r_x = \lambda(v-1)/(k-1)$, independent of x . Next double count incidences (member, committee): $vr = bk$, giving b . Integrality of r and b constrains (v, k, λ) : e.g. $\lambda = 1, k = 3$ (Steiner triple systems) forces $v \equiv 1$ or $3 \pmod{6}$. This computation is the opening move of design theory and a recurring CSIR objective item. $\blacksquare$

Worked Example 15.3 (*Counting incidences with an extremal twist*). (Olympiad staple.) In a class, every student knows exactly d others (acquaintance symmetric), and any two students have exactly one common acquaintance. Prove that some student knows everyone — or the friendship theorem’s conclusion: the configuration is a “windmill.” Prove at least the counting core: such a graph is d -regular and $n = d^2 - d + 1$.

Solution. Count paths of length 2 out of a vertex v : each of the $\binom{n-1}{1}$ other vertices u has exactly one common neighbour with v , so the number of 2-paths from v is $n-1$; but it also equals $d(d-1)$ (choose a neighbour, then a neighbour of it other than v — using the easily-derived fact that acquaintance-neighbourhoods intersect in one point, whence no double counting among these). So $n = d^2 - d + 1$. The full friendship theorem (Erdős–Rényi–Sós) then needs an eigenvalue argument — a famous instance where counting hands over to linear algebra, and a beautiful thing for an NBHM candidate to have read once. $\blacksquare$

15.3 Extremal counting: forbidding structures

Theorem 15.5 (Mantel). *A triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges, and the bound is attained by the balanced complete bipartite graph.*

Proof. For any edge uv of a triangle-free G , the neighbourhoods of u and v are disjoint: $\deg u + \deg v \leq n$. Sum over edges: $\sum_{uv \in E} (\deg u + \deg v) = \sum_v \deg^2 v \leq n|E|$. By Cauchy–Schwarz, $\sum_v \deg^2 v \geq \frac{(\sum \deg v)^2}{n} = \frac{4|E|^2}{n}$. Chain: $\frac{4|E|^2}{n} \leq n|E|$, so $|E| \leq n^2/4$. $\square$

The proof is pure counting plus one inequality — the standard texture of extremal combinatorics (Turán’s theorem generalizes; Kővári–Sós–Turán bounds C_4 -free graphs by double counting “cherries,” Exercise 6). ISI B.Math subjective and olympiads visit this ground constantly.

The Abstract Viewpoint • Counting as an existence oracle

This chapter’s methods prove theorems of the form “a configuration *exists*” or “*cannot exist*” without exhibiting or forbidding it constructively. The logical kernel: (1) *Averaging*: a random or average object achieves at least the mean — the embryo of the probabilistic method (Chapter 20), where “there is a box with two pigeons” matures into “a random colouring succeeds with positive probability.” (2) *Parity and invariants*: the handshake lemma’s “odd-degree vertices are even” is an invariant-based impossibility proof; olympiad colouring arguments are its descendants. (3) *Dimension/rank*: when pigeonhole is too weak, one counts *dimensions* — linear-algebraic double counting (as in the friendship theorem, or Fisher’s inequality $b \geq v$ proved by showing block-indicator vectors are independent). The ascent pigeonhole $\rightarrow$ counting in two ways $\rightarrow$ linear algebra $\rightarrow$ probability is the actual staircase of modern extremal combinatorics.

Deployment guide

Pigeonhole triggers: “show two objects agree/differ by,” “some subset sums,” “at least k of,” irrational-rotation and decimal problems (boxes: interval bins), geometry (boxes: a partition of the region — e.g. 5 points in a unit square, boxes: four quarter-squares, two points within distance $\frac{\sqrt{2}}{2}$). Double counting triggers: “every pair lies in exactly λ ,” regular structures, incidence totals, sums of degrees, averaging over rows vs columns of a 0-1 matrix. Write the incidence set explicitly — “count pairs (x, C) with $x \in C$ ” — and the two marginals; ambiguity dies there.

Exercises

- Seventeen points are placed in an equilateral triangle of side 4. Prove two are within distance 1. [ISI/OLYMPIAD]
- Show any $n + 1$ elements of $[2n]$ contain two consecutive; deduce again the coprime claim of Chapter 10. [ISI]
- From the decimal expansion pigeonhole, prove every rational has an eventually periodic expansion, and every integer m coprime to 10 divides some repunit $11 \cdots 1$. [ISI/CMI]
- In any 2-colouring of the edges of K_6 , show there are at least two monochromatic triangles. [Hint: count “bichromatic cherries” at each vertex] [OLYMPIAD]
- Prove Fisher’s inequality $b \geq v$ for $\lambda = 1$ designs by the rank argument sketched in the Abstract Viewpoint box. [NBHM]

6. (Kővári–Sós–Turán, baby case.) A graph on n vertices with no C_4 satisfies $|E| \leq \frac{1}{4}(1 + \sqrt{4n - 3})n$. *[Hint: double count paths of length 2: $\sum_v \binom{\deg v}{2} \leq \binom{n}{2}$; convexity]* [\[OLYMPIAD/NBHM\]](#)
7. A sequence of $n^2 + 1$ distinct reals is given. Deduce from Erdős–Szekeres a monotone subsequence of length $n + 1$, and show sharpness. [\[ISI/CMI\]](#)
8. (CSIR flavour.) Every point of the plane is coloured with one of three colours. Show some two points at distance exactly 1 share a colour. *[Hint: two unit equilateral triangles glued (the Moser spindle idea in miniature)]* [\[OLYMPIAD\]](#)

Group Actions, Burnside, and Pólya Enumeration

ELI5 • Counting bracelets without going mad

Paint the 4 beads of a square bracelet red or blue: $2^4 = 16$ paintings if the bracelet is nailed to a wall. But a bracelet turns in your hand: many paintings are secretly the same one. Dividing 16 by the 4 rotations gives 4 — wrong, because the all-red painting doesn't have four disguises; it has one. The honest rule: for each way of turning, count the paintings it fails to change, then *average*. Turn by nothing: all 16 unchanged. Quarter turns: only all-red, all-blue — 2 each. Half turn: 4. Average: $(16 + 2 + 4 + 2)/4 = 6$. Six genuinely different bracelets, and the average never lies.

16.1 Actions, orbits, stabilizers

Definition 16.1. An *action* of a group G on a set X is a map $G \times X \rightarrow X$, $(g, x) \mapsto g \cdot x$, with $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$. The *orbit* of x is $\text{Orb}(x) = \{g \cdot x : g \in G\}$; the *stabilizer* is $\text{Stab}(x) = \{g : g \cdot x = x\} \leq G$; the *fixed set* of g is $\text{Fix}(g) = \{x : g \cdot x = x\}$. Orbits partition X (the relation $x \sim g \cdot x$ is an equivalence), and X/G denotes the set of orbits.

Theorem 16.2 (Orbit–stabilizer). For finite G : $|\text{Orb}(x)| \cdot |\text{Stab}(x)| = |G|$.

Proof. The map $G \rightarrow \text{Orb}(x)$, $g \mapsto g \cdot x$, is surjective with fibres the left cosets of $\text{Stab}(x)$: $g \cdot x = h \cdot x \iff h^{-1}g \in \text{Stab}(x)$. Cosets partition G into equal blocks of size $|\text{Stab}(x)|$ (the map $s \mapsto gs$ bijects Stab with $g\text{Stab}$): quotient principle. $\square$

This is Chapter 4's diagnosis made precise: symmetric configurations (large stabilizers) sit in small orbits, so orbits vary in size, and $|X|/|G|$ fails. The repair is the celebrated average:

Theorem 16.3 (Burnside / Cauchy–Frobenius lemma). For a finite group G acting on a finite set X ,

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$

Proof. Double count the incidence set $I = \{(g, x) : g \cdot x = x\}$. Group-first: $|I| = \sum_g |\text{Fix}(g)|$. Element-first: $|I| = \sum_x |\text{Stab}(x)| = \sum_x \frac{|G|}{|\text{Orb}(x)|}$ by orbit–stabilizer. Within one orbit $\mathcal{O}$, $\sum_{x \in \mathcal{O}} \frac{1}{|\text{Orb}(x)|} = |\mathcal{O}| \cdot \frac{1}{|\mathcal{O}|} = 1$; so the element-first total is $|G| \cdot |X/G|$. Equate. $\square$

A proof of textbook perfection: orbit–stabilizer plus double counting, nothing else — every tool built in earlier chapters, none wasted.

Worked Example 16.1 (Necklace formula, redeemed). n beads on a circle, k colours, rotations only. The rotation by j positions fixes exactly the colourings constant on the cycles of that

rotation; the rotation by j has $\gcd(j, n)$ cycles, so $|\text{Fix}(r^j)| = k^{\gcd(j, n)}$, and

$$\#\text{necklaces} = \frac{1}{n} \sum_{j=0}^{n-1} k^{\gcd(j, n)} = \frac{1}{n} \sum_{d|n} \varphi\left(\frac{n}{d}\right) k^d,$$

grouping j by $\gcd(j, n) = d$ (there are $\varphi(n/d)$ such j). Chapter 4's preview and Chapter 14's Lyndon computation both land here — three independent derivations of one formula, which is how one knows a formula is true *and* understood.

Worked Example 16.2 (*Bracelets: the dihedral group*). Add reflections: $G = D_n$ of order $2n$. For n odd, each of the n reflections has $(n+1)/2$ cycles (one fixed bead, $(n-1)/2$ swapped pairs): fixed colourings $k^{(n+1)/2}$ each. For n even, $n/2$ reflections through two beads ($n/2 + 1$ cycles) and $n/2$ through two gaps ($n/2$ cycles). E.g. $n = 4, k = 2$: rotations contribute $16 + 2 + 4 + 2$; reflections: two bead-axes $2^3 = 8$ each, two gap-axes $2^2 = 4$ each; total $\frac{24+8+8+4+4}{8} = \frac{48}{8} = 6$ bracelets — matching an explicit listing.

16.2 Pólya's theorem: counting with weights

Burnside answers “how many”; Pólya answers “how many with exactly r red beads” — all r at once.

Definition 16.4. Let G act on a finite set D (of “positions”), each g decomposing D into $c_j(g)$ cycles of length j . The *cycle index* is the polynomial

$$Z_G(t_1, t_2, \dots) = \frac{1}{|G|} \sum_{g \in G} \prod_{j \geq 1} t_j^{c_j(g)}.$$

Theorem 16.5 (Pólya, 1937). *Colour the positions with colours $1, \dots, k$ carrying formal weights $w_1, \dots, w_k$; give a colouring the weight $\prod_{d \in D} w_{\chi(d)}$. Then the sum of weights over inequivalent colourings (orbits of G on colourings) is*

$$Z_G\left(\sum_i w_i, \sum_i w_i^2, \sum_i w_i^3, \dots\right).$$

Setting all $w_i = 1$ recovers Burnside's count $Z_G(k, k, \dots, k)$.

Proof. Apply the weighted Burnside lemma (same double count as Theorem 16.3, with each incidence (g, χ) weighted by $w(\chi)$; orbits have constant weight, so the element-first side totals $|G| \sum_{\text{orbits}} w$). It remains to compute the weighted fixed sum of one g : a colouring is fixed by g iff constant on each cycle of g ; choosing a colour independently per cycle, a j -cycle contributes its colour's weight j times, so the fixed-weight sum factors as $\prod_j (\sum_i w_i^j)^{c_j(g)}$. Average over g . $\square$

Worked Example 16.3 (*Weighted necklaces*). Necklaces of 6 beads, colours black/white, rotations only: how many with exactly 3 black?

Solution. $Z_{C_6} = \frac{1}{6}(t_1^6 + t_2^3 + 2t_3^2 + 2t_6)$ (rotations by 0; 3; 2, 4; 1, 5). Substitute $t_j \mapsto b^j + w^j$:

$$\frac{1}{6}[(b+w)^6 + (b^2+w^2)^3 + 2(b^3+w^3)^2 + 2(b^6+w^6)].$$

Coefficient of b^3w^3 : $\frac{1}{6}[20+0+2 \cdot 2+0] = \frac{24}{6} = 4$. Four necklaces with three black beads — and the full polynomial $b^6 + b^5w + 3b^4w^2 + 4b^3w^3 + 3b^2w^4 + bw^5 + w^6$ answers every “exactly r ” question

simultaneously: total 14 necklaces, matching $\frac{1}{6}(2^6 + 2^3 + 2 \cdot 2^2 + 2 \cdot 2 \cdot 2) = \frac{1}{6}(64 + 8 + 8 + 4) = 14$. ■

Worked Example 16.4 (*Cube colourings, CSIR/NBHM staple*). Count colourings of the 6 faces of a cube with k colours, up to rotation.

Solution. The rotation group has order 24; its action on faces has cycle census: identity t_1^6 (1 element); face-axis quarter turns $t_1^2 t_4$ (6); face-axis half turns $t_1^2 t_2^2$ (3); edge-axis half turns t_2^3 (6); vertex-axis third turns t_3^2 (8). Hence

$$\# = \frac{1}{24}(k^6 + 6k^3 + 3k^4 + 6k^3 + 8k^2) = \frac{k^6 + 3k^4 + 12k^3 + 8k^2}{24}.$$

$k = 2$: $\frac{64+48+96+32}{24} = 10$; $k = 3$: 57. The census itself — 1, 6, 3, 6, 8 summing to 24 — is worth memorizing; CSIR has asked the $k = 3$ number verbatim. ■

The Abstract Viewpoint • From Pólya to symmetric functions and species

The substitution $t_j \mapsto \sum_i w_i^j$ replaces cycle-index variables by *power sums* p_j : Pólya's theorem says orbit-weight enumerators are cycle indices evaluated at power sums — i.e., they live in the ring of symmetric functions, where Z_{S_n} evaluated at p_j is the complete homogeneous symmetric function h_n , and the identity $\sum_n Z_{S_n} x^n = \exp(\sum_j p_j x^j / j)$ is the cycle-type census of Chapter 3 exponentiated. Species theory absorbs all of this: the *cycle index series* Z_F of a species specializes to labelled counts ($p_1 = x$, rest 0) and unlabelled counts ($p_j = x^j$), and composition of species corresponds to plethysm of symmetric functions. Further up the same staircase: Burnside's lemma is the character-theoretic statement $|X/G| = \langle \chi_X, \mathbf{1} \rangle$ (the multiplicity of the trivial representation in the permutation representation), Pólya's refinement is its graded version, and de Bruijn's extension (a second group acting on colours) handles “colour-swappable” equivalences. Molecular isomer counting — Pólya's original motivation — and the enumeration of Boolean functions up to input permutation are the classical applications; both appear in CSIR-adjacent literature.

Burnside/Pólya drill discipline

(1) Identify D (positions) and G with its action on D — errors live here; write each group element's cycle type on D explicitly. (2) Small casework table: element class, class size, cycle monomial. (3) Burnside = substitute $t_j = k$; Pólya = substitute power sums. (4) Sanity: the orbit count must be an integer; the weighted enumerator must be symmetric in colours; setting $k = 1$ must give 1. Memorized cycle indices: C_n : $\frac{1}{n} \sum_{d|n} \varphi(d) t_d^{n/d}$; D_n (odd n): add $\frac{1}{2} t_1 t_2^{(n-1)/2}$; cube faces as computed; cube vertices: $\frac{1}{24}(t_1^8 + 9t_2^4 + 6t_4^2 + 8t_1^2 t_3^2)$.

Exercises

- Count the distinct dice: label cube faces 1–6, each label once, up to rotation. [*Hint: 6!/24; justify freeness*] [ISI/CMI]
- How many ways to 2-colour the vertices of a regular pentagon, up to rotation and reflection? Verify by listing. [ISI]
- Compute the number of necklaces of 8 beads with 3 colours, rotations only; then bracelets (add reflections). [CSIR]

4. Two chessboard corners: count 2-colourings of a 3×3 grid up to the 4 rotations. *[Hint: cycle types of C_4 on 9 cells: t_1^9 ; $t_1 t_4^2$ twice; $t_1 t_2^4$]* [CSIR/NBHM]
5. Count cube *edge* colourings with k colours up to rotation (derive the cycle index on 12 edges). [NBHM]
6. Prove: the number of orbits of G on $X \times X$ equals $\frac{1}{|G|} \sum_g |\text{Fix}(g)|^2$; deduce that a transitive action is 2-transitive iff $\frac{1}{|G|} \sum |\text{Fix}(g)|^2 = 2$ — Burnside as character inner product. [NBHM/CSIR]
7. How many chemically distinct ways to attach H/CH₃ groups at the 6 positions of a benzene ring (symmetry D_6)? Pólya's original playground. [CSIR]
8. (Olympiad.) Using Burnside on the cyclic group of prime order p acting on k -colourings of p beads, re-prove Fermat's little theorem $k^p \equiv k \pmod{p}$. [OLYMPIAD]

The Bijective Art: Involutions, Sign-Reversal, and Path Determinants

ELI5 • Proving two piles are equal without counting either

You have a pile of red tickets and a pile of blue tickets and want to show the piles are the same size. Don't count them — *pair* them: staple each red ticket to one blue ticket with none left over. Even better trick: if tickets come with $+$ and $-$ signs and you must show the signed total is zero, find a pairing that staples each $+$ ticket to a $-$ ticket. Cleverer still: allow a few unstapled survivors — then the signed total equals the survivors' total, which may be easy. Combinatorialists have turned this stapling into a fine art.

17.1 The involution principle

Definition 17.1. A *sign-reversing involution* on a signed set (X, ε) , $\varepsilon: X \rightarrow \{\pm 1\}$, is an involution $\iota: X \rightarrow X$ ($\iota^2 = \text{id}$) such that $\varepsilon(\iota(x)) = -\varepsilon(x)$ whenever $\iota(x) \neq x$. Then

$$\sum_{x \in X} \varepsilon(x) = \sum_{x \in \text{Fix}(\iota)} \varepsilon(x) :$$

non-fixed points cancel in pairs; the alternating sum localizes to the fixed set.

Every alternating identity in this book is an invitation to construct such an ι . Specimens:

Worked Example 17.1 (*The alternating row sum, bijectively*). $\sum_{S \subseteq [n]} (-1)^{|S|} = 0$ for $n \geq 1$: toggle element 1 ($\iota(S) = S \triangle \{1\}$) — sign-reversing, no fixed points. The partial-sum identity of Theorem 5.9 localizes the same involution to sets not containing... precisely: restrict to $|S| \leq k$; the toggle fails only when $S \ni 1$ and $|S| = k$ (adding 1 would overflow): fixed set of size $\binom{n-1}{k}$ and sign $(-1)^k$ — the identity falls out.

Worked Example 17.2 (*Vandermonde with signs; the “snake-oil” cancellation*). Rather than a zoo of signed Vandermonde variants, we prove the cleanest master specimen:

$$\sum_k (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^n [m = n].$$

Solution. The summand counts pairs (K, M) , $M \subseteq K \subseteq [n]$, $|M| = m$, signed by $(-1)^{|K|}$. Involution: toggle the largest element of $[n] \setminus M$ in or out of K . This is sign-reversing and defined unless $[n] \setminus M = \emptyset$, i.e. $M = [n]$, forcing $m = n$, $K = [n]$, sign $(-1)^n$. Identity proved with zero algebra — and the method, not the result, is the possession: ISI/CMI subjective papers reward exactly this craft. ■

17.2 The Lindström–Gessel–Viennot lemma

Signed cancellation reaches its summit in a determinant.

Theorem 17.2 (LGV). *In a locally finite acyclic directed graph, fix sources $A_1, \dots, A_n$ and sinks $B_1, \dots, B_n$, and let $e(A_i, B_j)$ be the number of directed paths $A_i \rightarrow B_j$. Suppose the configuration is non-permutable: any system of vertex-disjoint paths joining the A 's to the B 's necessarily connects $A_i \rightarrow B_i$ for every i . Then*

$$\det[e(A_i, B_j)]_{i,j=1}^n = \#\{\text{vertex-disjoint path systems } (A_i \rightarrow B_i)_{i=1}^n\}.$$

Proof. Expand the determinant: $\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_i e(A_i, B_{\sigma(i)})$ counts *all* path systems $(P_i: A_i \rightarrow B_{\sigma(i)})$, signed by $\text{sgn } \sigma$. Involution on systems with at least one crossing: locate the lexicographically first pair of paths sharing a vertex, and at their first common vertex *swap tails*. Tail-swapping is an involution, preserves the multiset of edges' start/end data except for transposing two targets — composing σ with a transposition — hence reverses sign. Fixed points: vertex-disjoint systems, which by non-permutability all carry $\sigma = \text{id}$, sign $+1$. $\square$

Worked Example 17.3 (*Counting non-crossing path families*). The number of pairs of lattice paths from $A_1 = (0, 1), A_2 = (1, 0)$ to $B_1 = (m, n+1) \dots$ take the classical clean case: vertex-disjoint pairs of monotone paths $(0, 0) \rightarrow (m, n)$ and $(1, -1) \rightarrow (m+1, n-1)$ number

$$\det \begin{pmatrix} \binom{m+n}{m} & \binom{m+n}{m+1} \\ \binom{m+n}{m-1} & \binom{m+n}{m} \end{pmatrix} = \binom{m+n}{m}^2 - \binom{m+n}{m+1} \binom{m+n}{m-1},$$

manifestly positive — LGV thus proves *log-concavity* of binomial rows: $\binom{m+n}{m}^2 \geq \binom{m+n}{m-1} \binom{m+n}{m+1}$, with the difference counting something. Determinantal positivity by counting is a modern signature move (totally positive matrices, Lindström's motivation).

Specializing LGV to staircase regions yields determinant formulas for counting families of non-crossing Dyck paths, rhombus tilings of hexagons (MacMahon's box formula), and — via the Jacobi–Trudi identity — the Schur-function determinants of symmetric function theory: one lemma irrigating three fields. NBHM interviews have touched LGV; olympiad problems secretly instantiate it.

17.3 Catalan structures, second visit: bijections with content

Worked Example 17.4 (*Non-crossing partitions and the Kreweras complement*). Show bijectively that non-crossing partitions of $[n]$ are counted by C_n , by constructing the standard bijection with Dyck paths.

Solution. Given a non-crossing partition, walk $i = 1..n$; at each i write U if i is the smallest element of its block, followed by... the crisp encoding: traverse $1, \dots, n$; at step i emit one U ; after step i emit one D for each block whose *maximum* is i ... this produces UD -words; verify balance and the ballot property from non-crossing-ness (a block cannot close before blocks opened inside it — exactly well-nested brackets). Inverse: match each D to its U ; the matching's nesting classes are the blocks. Non-crossing-ness of partitions *is* well-nestedness of brackets; the bijection is a change of notation once seen — the ideal state for a bijection. The lattice $NC(n)$ of non-crossing partitions carries the beautiful Kreweras complementation anti-automorphism; its Möbius function is $(-1)^{n-1} C_{n-1}$, dovetailing with Chapter 14's programme. $\blacksquare$

Worked Example 17.5 (*Cycle lemma in action, olympiad grade*). n people stand in a queue at a counter with an empty till; p hold Rs. 50 notes and $q = n - p$ hold Rs. 100 notes, $p \geq q$. Prove the probability that the till never fails (random order, people distinct) is $\frac{p-q+1}{p+1}$.

Solution. Orders correspond to words with p U's, q D's (note types), all orders equally likely, and within-type orderings cancel. Good words (every prefix has $\#U \geq \#D$) are ballot-with-ties: by reflection (Theorem 9.2's weak-inequality variant, Exercise 9.6), $\#good = \frac{p-q+1}{p+1} \binom{p+q}{q}$; divide by $\binom{p+q}{q}$. ■

The Abstract Viewpoint • What makes a bijection “good”?

Not all bijections are equal. The working aesthetic, distilled from practice: a good bijection is *local* (small changes to input make small changes to output), *equivariant* (commutes with natural symmetries — relabelling, complementation, rotation), and *statistic-preserving* (carries refined counts: the Dyck $\leftrightarrow$ tree bijection sends height to depth, peaks to leaves). Equivariance is the species-theoretic notion of naturality; statistic preservation is the q -analogue imperative of Chapter 19 — a bijection proving $f(n) = g(n)$ should ideally prove $f_q(n) = g_q(n)$. Garsia–Milne’s “involution principle” shows any two inclusion–exclusion–equal sets admit a bijection assembled from involutions — but such assembled bijections can be computationally monstrous, prompting the modern research question of when *efficient* bijections exist (e.g. the long-open search for a fully satisfying bijective proof of the enumeration of alternating sign matrices, resolved by Zeilberger algebraically but still bijectively mysterious). Counting, at research level, is the study of these correspondences themselves.

Bijjective toolkit inventory

Toggle involutions (parity identities); tail-swap (determinants, LGV); reflection (barrier counts); conjugation/transpose (partitions, Chapter 18); complementation (subsets, Kreweras); cycle lemma (ballot with ties); first-return decomposition (Catalan); Prüfer (trees); RSK (pairs of tableaux $\leftrightarrow$ permutations — know its statement: length of longest increasing subsequence = first row). In subjective papers, *state the map, prove well-definedness both ways, prove mutual inversion* — three paragraphs, no hand-waving; graders mark the middle paragraph hardest.

Exercises

- Construct a sign-reversing involution proving $\sum_k (-1)^k \binom{n}{k} \binom{2n-2k}{n-1} = 0$ for $n \geq 1$... if too stiff, prove instead $\sum_k (-1)^k \binom{n}{k} 2^{n-k} = 1$ by involution. [ISI]
- Prove the log-concavity $\binom{n}{k}^2 \geq \binom{n}{k-1} \binom{n}{k+1}$ directly by an injection between path pairs (no LGV determinant). [CMI/OLYMPIAD]
- Verify the LGV determinant numerically for $m = n = 2$ in the worked example by listing disjoint path pairs. [ISI]
- Establish the bijection between plane (ordered rooted) trees with n edges and Dyck paths of semilength n via the depth-first “glove” traversal, and identify which path statistic equals the number of leaves. [CMI/OLYMPIAD]
- Show that non-nesting and non-crossing perfect matchings of $[2n]$ are equinumerous (both C_n) but exhibit that the identity map is neither bijection; construct one honest bijection. [OLYMPIAD]

6. Use the cycle lemma to prove that the number of sequences of n $+1$'s and n -1 's whose partial sums are all ≥ 0 is C_n (yet again), this time with no reflection. [\[ISI\]](#)
7. (RSK literacy.) Accepting RSK, deduce $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$ where f^λ counts standard Young tableaux, and interpret $f^{(n,n)} = C_n$. [\[NBHM/CSIR\]](#)

Integer Partitions: Euler's Garden

ELI5 • Breaking a chocolate bar of 5 squares

How many ways to break the number 5 into positive pieces if order doesn't matter? 5; 4+1; 3+2; 3+1+1; 2+2+1; 2+1+1+1; 1+1+1+1+1: seven ways. Draw each as rows of dots — longest row on top — and you get little staircases called Ferrers diagrams. Flip a staircase along its diagonal and you get another valid staircase: this one flip, discovered centuries ago, proves theorems that pure calculation never could.

18.1 Definitions, diagrams, conjugation

Definition 18.1. A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ with $\sum \lambda_i = n$; write $\lambda \vdash n$, $|\lambda| = n$, $\ell(\lambda) = k$. Let $p(n) = \#\{\lambda \vdash n\}$ ($p(0) = 1$), $p_k(n)$ the count with exactly k parts. The *Ferrers diagram* of λ has λ_i left-justified cells in row i ; the *conjugate* λ' transposes the diagram ($\lambda'_j = \#\{i : \lambda_i \geq j\}$).

Theorem 18.2 (First conjugation harvest). *Conjugation is an involution on partitions of n exchanging (number of parts) $\leftrightarrow$ (largest part). Hence: the number of partitions of n into at most k parts equals the number into parts of size at most k ; and $p_k(n) = \#\{\lambda \vdash n : \lambda_1 = k\}$.*

Proof. Transposing swaps rows and columns; row count and first-column height exchange with column count and first-row width. All claims are re-readings of this exchange. $\square$

This closes the twelfold way: identical objects into k identical boxes, none empty, is $p_k(n)$; at most k boxes is $p_{\leq k}(n)$, now also readable as bounded-part-size partitions.

Theorem 18.3 (Partition recurrence). $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.

Proof. Split partitions with exactly k parts by whether the smallest part is 1: if yes, delete it ($p_{k-1}(n-1)$); if no, subtract 1 from every part — delete the first column — ($p_k(n-k)$). Both maps are bijections onto their targets. $\square$

18.2 Generating functions of Euler

Theorem 18.4.

$$\sum_{n \geq 0} p(n) x^n = \prod_{j \geq 1} \frac{1}{1 - x^j}, \quad \sum_n p_{\leq k}(n) x^n = \prod_{j=1}^k \frac{1}{1 - x^j},$$

and partitions into distinct parts, respectively odd parts, have OGFs $\prod_j (1 + x^j)$ and $\prod_{j \text{ odd}} \frac{1}{1 - x^j}$.

Proof. A partition is a multiset of part-sizes: for each size j choose a multiplicity $m_j \geq 0$, contributing x^{jm_j} ; the factor for size j is $\sum_{m \geq 0} x^{jm} = \frac{1}{1 - x^j}$, and independence multiplies factors (the infinite product is legitimate formally since $[x^n]$ depends on finitely many factors). Distinct

parts: $m_j \in \{0, 1\}$: factor $1 + x^j$. Bounded largest part $\leq k$: keep factors $j \leq k$; combine with conjugation for the “at most k parts” claim. $\square$

Theorem 18.5 (Euler). *The number of partitions of n into distinct parts equals the number into odd parts.*

Proof 1 (algebra, three lines). $\prod_j (1 + x^j) = \prod_j \frac{1 - x^{2j}}{1 - x^j}$; cancel every even-index denominator factor against the numerators, leaving $\prod_{j \text{ odd}} \frac{1}{1 - x^j}$. $\square$

Proof 2 (Glaisher’s bijection). From odd parts to distinct parts: given odd parts with multiplicities, write each multiplicity in binary, $m = \sum 2^a$, and replace m copies of the odd part u by the parts $\{2^a u\}$ — distinct, since every positive integer is uniquely $2^a u$ with u odd. That uniqueness is also the inverse map: split each distinct part into its dyadic and odd factors and regroup. A model bijection: it is the fundamental theorem of arithmetic, localized at 2, wearing combinatorial clothes. $\square$

Worked Example 18.1 (ISI/CMI drill). Compute $p(n)$ for $n \leq 8$ and $p_3(8)$; verify $p_3(8) = p_2(7) + p_3(5)$.

Solution. $p(0), \dots, p(8)$ read 1, 1, 2, 3, 5, 7, 11, 15, 22. Partitions of 8 into exactly 3 parts: $6+1+1$, $5+2+1$, $4+3+1$, $4+2+2$, $3+3+2$: $p_3(8) = 5$; and $p_2(7) = 3$ ($6+1$, $5+2$, $4+3$), $p_3(5) = 2$ ($3+1+1$, $2+2+1$): $3 + 2 = 5$. $\checkmark$ $\blacksquare$

18.3 Euler’s pentagonal number theorem

Theorem 18.6 (Pentagonal number theorem).

$$\prod_{j \geq 1} (1 - x^j) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(3m-1)/2} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Proof (Franklin’s involution, 1881). Expanding the product, $[x^n]$ is $\sum_{\lambda} (-1)^{\ell(\lambda)}$ over partitions $\lambda \vdash n$ into *distinct* parts. Define Franklin’s map: let $s(\lambda)$ be the smallest part and $r(\lambda)$ the length of the “staircase run” at the top (maximal r with $\lambda_1, \lambda_2, \dots, \lambda_r$ consecutive descending integers). If $s \leq r$, remove the smallest part and add 1 to each of the top s parts; if $s > r$, subtract 1 from each of the top r parts and append a new smallest part r . Each operation changes ℓ by ∓ 1 — sign-reversing — and they are mutually inverse where defined. The construction fails exactly when the staircase run collides with the smallest part (the diagram is a perfect “pentagonal” trapezoid): for $\ell = m$ these exceptional partitions have $n = m(3m \pm 1)/2$, one for each such n , of sign $(-1)^m$. The alternating sum localizes to these survivors. $\square$

Franklin’s argument is the crown jewel of Chapter 17’s involution principle and among the most beautiful proofs in mathematics — ISI/CMI interviewers and olympiad camps present it as a set piece.

Corollary 18.7 (Recursive computation of $p(n)$). $p(n) = \sum_{m \geq 1} (-1)^{m-1} \left[p\left(n - \frac{m(3m-1)}{2}\right) + p\left(n - \frac{m(3m+1)}{2}\right) \right]$, terms with negative arguments discarded. E.g. $p(9) = p(8) + p(7) - p(4) - p(2) = 22 + 15 - 5 - 2 = 30$.

Proof. Multiply $\sum p(n)x^n$ by the pentagonal series and equate coefficients of x^n in 1: convolution. $\square$

The Abstract Viewpoint • From Euler to modular forms

The partition generating function is, up to the factor $x^{1/24}$, the reciprocal of Dedekind's eta function $\eta(\tau) = q^{1/24} \prod (1 - q^n)$, a modular form of weight $\frac{1}{2}$ — partitions are the Fourier coefficients of an automorphic object. This is not decoration: modularity is *why* Hardy–Ramanujan could prove

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad (n \rightarrow \infty)$$

by the circle method (later perfected by Rademacher into an exact convergent series), and why Ramanujan's congruences $p(5n+4) \equiv 0 \pmod{5}$, $p(7n+5) \equiv 0 \pmod{7}$, $p(11n+6) \equiv 0 \pmod{11}$ hold — statements about staircase diagrams governed by Hecke operators. The combinatorial counterpart of the mod-5 congruence, Dyson's *rank* and the Andrews–Garvan *crank*, restore bijective meaning to the congruences: statistics equidistributing partitions of $5n+4$ into five equal classes. Meanwhile Jacobi's triple product identity $\prod (1 - q^{2n})(1 + zq^{2n-1})(1 + z^{-1}q^{2n-1}) = \sum_{m \in \mathbb{Z}} z^m q^{m^2}$ generalizes the pentagonal theorem and feeds theta functions, sums of squares, and affine Lie algebra characters (Macdonald identities). Few subjects display so vividly that “elementary” counting sits atop an ocean.

Partition problems in examinations

CSIR/NBHM: compute small $p(n)$, $p_k(n)$ via the recurrence; recognize odd=distinct; identify GFs of restricted partitions instantly (parts $\equiv r \pmod{m}$): keep those factors; parts appearing $< k$ times: factor $\frac{1-x^{kj}}{1-x^j}$ — whence Glaisher's general theorem: parts not divisible by $k \leftrightarrow$ parts repeating $< k$ times). ISI/CMI subjective: conjugation arguments and small bijections; self-conjugate partitions $\leftrightarrow$ distinct odd parts (peel L-shaped hooks) is the single most-asked specimen. Olympiad: Franklin-style involutions and staircase decompositions. Know $p(n)$, $n \leq 10$: 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42.

Exercises

1. List all partitions of 7 and verify $p(7) = 15$; mark the self-conjugate ones. [ISI]
2. Prove bijectively: self-conjugate partitions of $n \leftrightarrow$ partitions of n into distinct odd parts. [ISI/CMI]
3. Write the OGF for partitions of n into parts $\equiv \pm 1 \pmod{5}$ (the Rogers–Ramanujan product); state — as cultural literacy — the Rogers–Ramanujan theorem equating them with partitions whose parts differ by ≥ 2 . [NBHM]
4. Show that the number of partitions of n in which each part appears at most twice equals the number into parts not divisible by 3. [CMI]
5. Use Corollary 18.7 to compute $p(10), \dots, p(15)$ (42, 56, 77, 101, 135, 176). [CSIR]
6. Prove $p_k(n) \leq \binom{n-1}{k-1}$ and $p(n) \leq p(n-1) + p(n-2)$ is false in general — instead prove the true monotone bound $p(n) \leq 2p(n-1)$ and the superadditivity $p(a)p(b) \leq p(a+b)$. [OLYMPIAD]
7. Durfee squares: show every $\lambda \vdash n$ has a unique largest $d \times d$ square in its diagram, and derive $\sum_n p(n)x^n = \sum_{d \geq 0} \frac{x^{d^2}}{\prod_{j=1}^d (1-x^j)^2}$. [NBHM/OLYMPIAD]

q -Analogues: Counting with a Memory

ELI5 • Counting that remembers how tangled things are

Ordinary counting answers “how many?” A q -count answers “how many, and how tangled is each one?” Give every arrangement a tangle-score — say, the number of pairs standing in the wrong order — and instead of adding 1 per arrangement, add q^{score} . The total is a polynomial in q . Set $q = 1$ and the scores are forgotten: ordinary counting returns. Keep q and you own strictly more information — and, astonishingly, the familiar formulas ($n!$, $\binom{n}{r}$, Pascal’s rule) all survive with q ’s woven in.

19.1 q -numbers, q -factorials, inversions

Definition 19.1. For $n \geq 0$ set $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}$, $[n]_q! = [1]_q [2]_q \cdots [n]_q$,

$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!}$ (the *Gaussian binomial*). At $q = 1$ these are n , $n!$, $\binom{n}{r}$.

Definition 19.2. An *inversion* of a permutation σ of $[n]$ is a pair $i < j$ with $\sigma(i) > \sigma(j)$; $\text{inv}(\sigma)$ is their number.

Theorem 19.3 (Rodrigues, 1839). $\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]_q!$.

Proof. Build σ by inserting n into a permutation of $[n-1]$: inserting into position $n-i$ from the right creates exactly i new inversions, $i = 0, \dots, n-1$, contributing the factor $1 + q + \cdots + q^{n-1} = [n]_q$ independently of the base permutation’s inversions. Induct: the generating polynomial factors as $[n]_q [n-1]_q \cdots [1]_q$. $\square$

The proof upgrades the product rule to a *weighted* product rule: independent stages multiply weight-generating polynomials. All of Part I survives such upgrading — the deepest sense in which the q -world is not a novelty but a refinement.

Theorem 19.4 (Gaussian binomials count by weight).

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \sum_{\lambda \subseteq r \times (n-r)} q^{|\lambda|} = \sum_w q^{\text{inv}(w)},$$

where the middle sum runs over partitions whose Ferrers diagram fits in an $r \times (n-r)$ box, and the right sum over binary words w with r ones and $n-r$ zeros, inv counting pairs (1 before 0). In particular $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is a polynomial in q with nonnegative integer coefficients, of degree $r(n-r)$, with symmetric (palindromic) coefficient sequence.

Proof. Words and box-partitions correspond: read the word as a lattice path (1 = up, 0 = right) tracing the boundary of a Ferrers shape inside the box; inversions = cells above the path = $|\lambda|$. It

remains to identify the word-inversion polynomial with the ratio of q -factorials. Use the q -Pascal recurrences, proved by classifying the last letter: a word ending in 0 loses nothing ($\begin{bmatrix} n-1 \\ r \end{bmatrix}_q$); ending in 1, its final 1 sits after all $n-r$ zeros. . . carefully: removing a final 0 leaves inversions unchanged; removing a final 1 removes $n-r$ inversions of that 1 with nothing — recount: a final 1 forms no inversion (nothing after it), but contributes. . . cleanest: classify the *first* letter. First letter 1: it inverts with all $n-r$ zeros: factor $q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q$. First letter 0: no new inversions: $\begin{bmatrix} n-1 \\ r \end{bmatrix}_q$. Hence

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} n-1 \\ r \end{bmatrix}_q + q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q,$$

and the ratio-of- q -factorials satisfies the same recurrence and initial conditions (direct algebra with $[n]_q = [n-1]_q + q^{n-1}$): induction finishes. Degree: the all-inversions word $1^r 0^{n-r}$ has $r(n-r)$; symmetry: reverse the word (complement the partition in the box), an involution sending $|\lambda|$ to $r(n-r) - |\lambda|$. $\square$

The companion recurrence $\begin{bmatrix} n \\ r \end{bmatrix}_q = q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}_q + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q$ (classify the *last* letter properly) shows q -Pascal is genuinely two rules; at $q = 1$ they collapse into one — the q -world resolves degeneracies.

19.2 Subspaces over finite fields

Theorem 19.5. *Let q be a prime power. The number of r -dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ r \end{bmatrix}_q$ evaluated at that q .*

Proof. Ordered linearly independent r -tuples in $\mathbb{F}_q^n$: choose $v_1 \neq 0$ ($q^n - 1$ ways), $v_2 \notin \langle v_1 \rangle$ ($q^n - q$), . . . : $\prod_{i=0}^{r-1} (q^n - q^i)$ tuples. Each r -subspace contains exactly $\prod_{i=0}^{r-1} (q^r - q^i)$ such tuples (same count inside $\mathbb{F}_q^r$): uniform fibres; divide:

$$\frac{\prod_{i=0}^{r-1} (q^n - q^i)}{\prod_{i=0}^{r-1} (q^r - q^i)} = \prod_{i=0}^{r-1} \frac{q^{n-i} - 1}{q^{r-i} - 1} = \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

$\square$

So the Gaussian binomial has (at least) three lives: inversion-weighted words, box-bounded partitions, and subspace counts — with the slogan *sets are to subsets as $\mathbb{F}_q$ -spaces are to subspaces; classical combinatorics is the $q \rightarrow 1$ limit, “linear algebra over the field with one element.”* Identities transfer: q -Vandermonde $\begin{bmatrix} m+n \\ r \end{bmatrix}_q = \sum_k q^{(m-k)(r-k)} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ r-k \end{bmatrix}_q$ counts subspaces by intersection data; q -row-sums count all subspaces (Galois numbers); the q -binomial theorem

$$\prod_{i=0}^{n-1} (1 + q^i t) = \sum_{r=0}^n q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q t^r$$

is proved by classifying which factors donate t and weighing the crossings — Exercise 3 walks it.

Worked Example 19.1 (*CSIR pattern*). How many 2-dimensional subspaces does $\mathbb{F}_2^4$ have? How many lines (1-dimensional subspaces) does $\mathbb{F}_3^3$ have?

Solution. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q=2} = \frac{(2^4-1)(2^4-2)}{(2^2-1)(2^2-2)} = \frac{15 \cdot 14}{3 \cdot 2} = 35$. Lines in $\mathbb{F}_3^3$: $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q=3} = [3]_3 = 1 + 3 + 9 = 13$ — the 13 points of the projective plane $\text{PG}(2, 3)$. $\blacksquare$

Worked Example 19.2 (*q -counting a JEE object, ISI style*). Compute $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$ as a polynomial and interpret each coefficient.

Solution. Partitions in a 2×2 box: $\emptyset; 1; 2, 1+1; 2+1; 2+2$ — weights $q^0, q^1, 2q^2, q^3, q^4$: $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$. At $q = 1$: $6 = \binom{4}{2}$; the coefficient sequence $1, 1, 2, 1, 1$ is symmetric and unimodal — unimodality of Gaussian coefficients is a genuine theorem (Sylvester 1878; O'Hara 1990 combinatorially), false to prove by the naive ratio test that worked at $q = 1$: refinement has a price. ■

The Abstract Viewpoint • Why q ?

Four converging answers. **Statistics:** q tracks a combinatorial statistic (inversions, major index — equidistributed by MacMahon's theorem, founding the field of “permutation statistics”). **Finite fields:** q is literally a prime power; identities become geometry over $\mathbb{F}_q$, and Weil-philosophy conjectures translate point counts into topology. **Quantum:** $[n]_q$ are the characters that deform $\mathfrak{sl}_2$ into the quantum group $U_q(\mathfrak{sl}_2)$; Gaussian binomials are dimensions in its representation theory, and their palindromicity is a shadow of Poincaré duality / hard Lefschetz — the modern proofs of unimodality run through exactly this. **Analysis:** q -series (q -binomial theorem, Jacobi triple product, Rogers–Ramanujan) form Ramanujan's home terrain; Chapter 18's partition theory is the $n \rightarrow \infty$ limit of this chapter ($\lim_n \begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{i \leq r} \frac{1}{1-q^i}$: the box walls recede and bounded partitions become all partitions). A student who internalizes “every counting formula wants to be a polynomial in q ” has crossed from problem-solving into structural combinatorics.

q -literacy for examinations

CSIR/NBHM directly ask: subspace counts over small $\mathbb{F}_q$ (compute via the product formula, never expand blindly); the order of $GL_n(\mathbb{F}_q)$: $\prod_{i=0}^{n-1} (q^n - q^i)$ (the independent-tuple count above with $r = n$); and hence Sylow-flavoured factorizations. ISI/CMI: inversion counts, the polynomial $\begin{bmatrix} n \\ r \end{bmatrix}_q$ for small cases, symmetric unimodal recognition. Olympiad: inversion parity = permutation sign; inv-based induction. Values to know: $|GL_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$; $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = [n]_q$; $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$.

Exercises

1. Prove the second q -Pascal rule $\begin{bmatrix} n \\ r \end{bmatrix}_q = q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}_q + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q$ by the last-letter classification, tracking inversions carefully. [ISI/CMI]
2. Show $\text{inv}(\sigma)$ and $\text{inv}(\sigma^{-1})$ are equal, and that $(-1)^{\text{inv}}$ is multiplicative — the sign homomorphism. [CSIR]
3. Prove the q -binomial theorem $\prod_{i=0}^{n-1} (1 + q^i t) = \sum_r q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q t^r$ by induction, and deduce at $t = -1$ a q -analogue of the alternating row sum. [NBHM]
4. Count the subspaces of all dimensions of $\mathbb{F}_2^3$ and draw the subspace lattice; compare with the Boolean lattice $2^{[3]}$. [CSIR]
5. Compute $|GL_3(\mathbb{F}_2)| = 168$ and identify the group (simple, $\cong PSL_2(\mathbb{F}_7)$ — statement only). [CSIR/NBHM]
6. Show $\lim_{q \rightarrow 1} \begin{bmatrix} n \\ r \end{bmatrix}_q = \binom{n}{r}$ by l'Hôpital-free algebra (factor each $\frac{1-q^a}{1-q^b}$). [ISI]
7. Verify MacMahon's equidistribution for S_3 : the generating polynomials of inv and of the major index $\text{maj}(\sigma) = \sum_{\sigma(i) > \sigma(i+1)} i$ coincide (both $= [3]_q! = 1 + 2q + 2q^2 + q^3$). [OLYMPIAD/NBHM]

The Probabilistic Method and a Closing Synthesis

ELI5 • Proving a needle exists without finding it

You suspect a haystack contains at least one needle. Strange but true strategy: grab a handful *at random* and compute the *average* number of needles per handful. If the average is more than zero, some handful contains a needle — so a needle exists — even though you never located one. Averages certify existence. That modest thought, pushed hard by Erdős, became one of the most powerful ideas of twentieth-century mathematics.

20.1 Expectation as a counting weapon

For a finite probability space (here always uniform over a finite set of configurations), the expectation of $X = \sum_i X_i$ obeys *linearity*, $\mathbb{E}[X] = \sum_i \mathbb{E}[X_i]$, with no independence hypothesis — the probabilistic transcription of double counting. Two principles do most of the work:

The two pillars

Existence by averaging: some configuration achieves $X \geq \mathbb{E}[X]$, and some achieves $X \leq \mathbb{E}[X]$. **Existence by positive probability:** if $\Pr[\text{bad}] < 1$ — e.g. by the union bound $\Pr[\bigcup B_i] \leq \sum \Pr[B_i]$ — a good configuration exists. Both are Chapter 15’s pigeonhole, renormalized.

Theorem 20.1 (Erdős 1947: Ramsey lower bound). *If $\binom{n}{2} 2^{1-\binom{k}{2}} < 1$, then there exists a 2-colouring of the edges of K_n with no monochromatic K_k . Consequently the Ramsey number satisfies $R(k, k) > 2^{k/2}$ for $k \geq 3$.*

Proof. Colour each edge independently red/blue with probability $\frac{1}{2}$. For a fixed k -set S , $\Pr[S \text{ monochromatic}] = 2 \cdot 2^{-\binom{k}{2}}$. Union bound over all $\binom{n}{k} \leq \binom{n}{2}$. — bound cleanly: $\Pr[\exists \text{ mono } K_k] \leq \binom{n}{k} 2^{1-\binom{k}{2}}$; if this is < 1 , some colouring avoids all monochromatic K_k . For $n = 2^{k/2}$: $\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} 2^{1-\binom{k}{2}} = \frac{2^{k^2/2+1}}{k! 2^{k(k-1)/2}} = \frac{2^{k/2+1}}{k!} < 1$ for $k \geq 3$. $\square$

The proof exhibits nothing — no colouring is constructed, and to this day no explicit construction matches the random one’s quality: a mathematical and philosophical landmark. (The statement as displayed with $\binom{n}{k}$ is the honest hypothesis; the display’s first inequality with $\binom{n}{2}$ is a typo guard — always re-derive rather than recall.)

Worked Example 20.1 (*Averaging, olympiad grade*). Every tournament (complete directed graph) on n players contains a Hamiltonian directed path.

Solution. Order the players uniformly at random; let X count consecutive pairs in the order where the earlier beats the later. Each of the $n-1$ consecutive pairs is correctly directed with probability $\frac{1}{2}$: $\mathbb{E}[X] = \frac{n-1}{2}$. So some order has $X \geq \lceil (n-1)/2 \rceil$ — which proves a long path exists but not yet

Hamiltonicity. The clean Hamiltonicity proof is inductive/extremal (insert the new player into the longest path at the first place he wins); the probabilistic estimate, due to Szele 1943 — the method’s true birth — shows more: some tournament has at least $n!/2^{n-1}$ Hamiltonian paths, by computing the expected number over random tournaments. Distinguishing what averaging proves from what it suggests is part of the training. ■

Worked Example 20.2 (*Sum-free sets, Erdős 1965*). Every set A of n nonzero integers contains a sum-free subset (no $x + y = z$) of size $> n/3$.

Solution. Choose a prime $p = 3k + 2$ exceeding $2 \max|a|$ and work in $\mathbb{Z}/p$. The middle third $M = \{k + 1, \dots, 2k + 1\}$ of $\mathbb{Z}/p^*$ is sum-free mod p and has size $k + 1 > \frac{p-1}{3}$. Pick t uniform in $\mathbb{Z}/p^*$ and let $A_t = \{a \in A : ta \bmod p \in M\}$: each a lands in M with probability $\frac{k+1}{p-1} > \frac{1}{3}$, so $\mathbb{E}|A_t| > \frac{n}{3}$; some t achieves it, and A_t is sum-free (a relation $x + y = z$ in A_t would give one in $M \bmod p$). Averaging over a cleverly chosen group of dilations: the designed-boxes art of Chapter 15, now with probability doing the bookkeeping. ■

20.2 Second moments, concentration, and the frontier (a map)

Beyond expectation: variance (*second moment method*) shows random quantities cluster near their mean, proving *typical* structure (thresholds in random graphs $G(n, p)$: e.g. connectivity threshold at $p = \frac{\ln n}{n}$); the Lovász Local Lemma replaces the union bound when bad events are numerous but weakly dependent; concentration inequalities (Chernoff, Azuma, Talagrand) quantify “almost all.” These belong to a successor course (Alon–Spencer), but their existence reframes everything earlier: enumeration (how many), extremal combinatorics (how large), and probabilistic combinatorics (how typical) are one subject seen at three magnifications — and CSIR increasingly samples all three.

20.3 Synthesis: one problem, every tool

We close by solving a single problem five ways — a deliberate exhibition of the book’s architecture.

Worked Example 20.3 (*Pentathlon*). Count the binary strings of length n with no two consecutive 1s.

Solution. **(1) Recurrence** (Ch. 2/13): $a_n = a_{n-1} + a_{n-2}$, $a_n = F_{n+2}$. **(2) Direct combination count** (Ch. 6): choose r non-adjacent positions: $\sum_r \binom{n-r+1}{r}$ — whence the identity $\sum_r \binom{n-r+1}{r} = F_{n+2}$, a diagonal of Pascal. **(3) Transfer matrix** (Ch. 13): $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, count = $\mathbf{s}^T T^{n-1} \mathbf{f}$; eigenvalues φ, ψ give Binet. **(4) Generating function** (Ch. 12): blocks 0 and 01 compose freely with a possible trailing 1: $A(x) = \frac{1+x}{1-x-x^2}$; partial fractions give the closed form. **(5) Bijection** (Ch. 17): append 01-blocks $\leftrightarrow$ tilings of a $1 \times (n + 1)$ strip by squares and dominoes — the standard Fibonacci model, from which identities like $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$ drop out by cutting tilings at position m . Five proofs, five chapters, one number. The examination hall wants any one; mathematical maturity wants the dictionary between all five. ■

The Abstract Viewpoint • The staircase, seen from the top

Retrace the ascent. Counting began as bijection (Ch. 1) with two composition laws (Ch. 2); the classical formulas were quotients of free actions (Ch. 3–6); alternating sums were Möbius functions of the Boolean lattice (Ch. 7, 14); arithmetic of counts reflected p -adic structure (Ch. 8); barrier problems yielded to reflection and sign-reversal (Ch. 9, 17); the

twelfold way organized quotients by symmetric groups (Ch. 11); generating functions decategorified species (Ch. 12–13); genuine symmetry required averaging over groups (Ch. 4, 16); partitions opened onto modular forms (Ch. 18); q revealed every count as the shadow of a graded, geometric, or quantum object (Ch. 19); and probability turned averages into existence proofs (Ch. 15, 20). The pattern throughout: *a number is the decategorified trace of a structure; the structure explains the number*. Whoever carries this principle into an examination hall — or a research seminar — no longer merely counts. They see.

Final drill doctrine, all examinations

Daily: 10 mixed objective items at EAMCET/JEE-Main tempo (formula reflexes decay in weeks). **Alternate days:** one JEE-Advanced chain and one ISI/CMI subjective write-up in full sentences. **Weekly:** one derivation reproduced from memory — rotate: derangements; surjections; Burnside; pentagonal theorem sketch; q -Pascal; Erdős Ramsey bound. **Before any paper:** rewrite the selection table, the twelfold way, the transform list (glue/gap/complement/anchor/reflect/shift/slack), and the landmark constants ($D_n, C_n, B_n, p(n)$, cycle indices) on one sheet from memory. The sheet, not this book, enters the hall in your head.

Exercises

1. Random permutation of $[n]$: compute $\mathbb{E}[\text{fixed points}] = 1$ and $\text{Var} = 1$ by indicators; compare with the Poisson limit of Chapter 7. [CSIR]
2. Show every graph with m edges contains a bipartite subgraph with $\geq m/2$ edges (random vertex bipartition; averaging). [OLYMPIAD/NBHM]
3. Prove: any k -uniform hypergraph with fewer than 2^{k-1} edges is 2-colourable with no monochromatic edge. [NBHM]
4. In a random 2-colouring of $[n]$, bound the expected number of monochromatic 3-term APs and deduce a colouring with at most that many. [OLYMPIAD]
5. Complete the Fibonacci pentathlon: prove $F_{m+n} = F_m F_{n+1} + F_{m-1} F_n$ by cutting tilings, and re-prove it by $T^{m+n} = T^m T^n$. [ISI/CMI]
6. (Szele.) Compute the expected number of Hamiltonian paths in a random tournament and conclude some tournament has $\geq n!/2^{n-1}$. [NBHM]
7. Write, in at most two pages and from memory, the five solutions of the Pentathlon problem. This is the book's final examination. [ALL]

Tables, Landmark Numbers, and the Formula Sheet

This appendix is the book compressed to its skeleton. Nothing here should be *memorised before* it is understood; everything here should be *retrievable in seconds after* it is understood. The intended use is revision: in the final week before an examination, a reader should be able to reconstruct the proof of every line in these tables from the chapter cited beside it.

A.1 The Core Formula Sheet

Object counted	Formula	Ch.
Words of length r , alphabet n	n^r	2
Injections / arrangements	$n^{\underline{r}} = \frac{n!}{(n-r)!}$	3
Permutations of n distinct objects	$n!$	3
Multiset permutations	$\frac{n!}{n_1! n_2! \cdots n_k!}$	3
Circular arrangements	$(n-1)!$; garlands $\frac{1}{2}(n-1)!$	4
r -subsets of $[n]$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	5
Weak compositions: $x_1 + \cdots + x_k = n, x_i \geq 0$	$\binom{n+k-1}{k-1}$	6
Positive compositions: $x_i \geq 1$	$\binom{n-1}{k-1}$	6
Subsets of $[n]$, size r , no two adjacent	$\binom{n-r+1}{r}$	6
Surjections $[n] \twoheadrightarrow [k]$	$\sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n = k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	7, 11
Derangements	$D_n = n! \sum_{j=0}^n \frac{(-1)^j}{j!}$	7
Exactly m fixed points	$\binom{n}{m} D_{n-m}$	7, 10
Lattice paths $(0,0) \rightarrow (m,n)$	$\binom{m+n}{m}$	9
Dyck paths / balanced sequences	$C_n = \frac{1}{n+1} \binom{2n}{n}$	9
Necklaces, n beads, k colours	$\frac{1}{n} \sum_{d n} \varphi(d) k^{n/d}$	4, 16
Set partitions into k blocks	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$; all: B_n	11
Permutations with k cycles	$\left[\begin{matrix} n \\ k \end{matrix} \right]$	11
Integer partitions of n	$p(n)$; GF $\prod_{k \geq 1} (1-x^k)^{-1}$	18
r -subspaces of $\mathbb{F}_q^n$	$\left[\begin{matrix} n \\ r \end{matrix} \right]_q$	19

A.2 The Identity Calculus at a Glance

Name	Statement	Ch.
Symmetry	$\binom{n}{r} = \binom{n}{n-r}$	5
Pascal	$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$	5
Committee-chair	$r\binom{n}{r} = n\binom{n-1}{r-1}$	5
Row sum	$\sum_r \binom{n}{r} = 2^n$	5
Alternating row sum	$\sum_r (-1)^r \binom{n}{r} = [n=0]$	5
Vandermonde	$\sum_k \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$	5
Hockey stick	$\sum_{j=r}^n \binom{j}{r} = \binom{n+1}{r+1}$	5
Binomial theorem	$(x+y)^n = \sum_r \binom{n}{r} x^r y^{n-r}$	5
Stirling basis change	$x^n = \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^{\overline{k}}; \quad x^{\overline{n}} = \sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$	11
q -Pascal	$\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]_q = \left[\begin{smallmatrix} n-1 \\ r-1 \end{smallmatrix} \right]_q + q^r \left[\begin{smallmatrix} n-1 \\ r \end{smallmatrix} \right]_q$	19
Pentagonal numbers	$\prod_{k \geq 1} (1 - x^k) = \sum_{j \in \mathbb{Z}} (-1)^j x^{j(3j-1)/2}$	18

A.3 Landmark Numbers

An examination hall is not the place to recompute D_5 . The tables below list the values that recur so often across EAMCET, JEE, ISI, CMI and olympiad papers that they deserve residence in long-term memory.

Factorials and central binomials

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\binom{2n}{n}$	2	6	20	70	252	924	3432	12870	48620	184756

Derangements, Catalan, Bell

n	0	1	2	3	4	5	6	7
D_n	1	0	1	2	9	44	265	1854
C_n	1	1	2	5	14	42	132	429
B_n	1	1	2	5	15	52	203	877

Note the trap lovingly set by the subject itself: $C_3 = B_3 = 5$ and $C_2 = B_2 = 2$, after which the sequences part company forever.

Partition numbers

n	0	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	1	2	3	5	7	11	15	22	30	42

Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$

$n \backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ (unsigned)

$n \backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	2	3	1			
4	6	11	6	1		
5	24	50	35	10	1	
6	120	274	225	85	15	1

A.4 Cycle Indices for Pólya Counting

For a colour set of size k , substitute $p_i \mapsto k$ throughout to count orbits; substitute $p_i \mapsto x_1^i + x_2^i + \dots$ for the full weighted inventory (Chapter 16).

$$Z(C_n) = \frac{1}{n} \sum_{d|n} \varphi(d) p_d^{n/d} \quad (\text{cyclic: necklaces})$$

$$Z(D_n) = \frac{1}{2} Z(C_n) + \begin{cases} \frac{1}{2} p_1 p_2^{(n-1)/2}, & n \text{ odd,} \\ \frac{1}{4} \left(p_2^{n/2} + p_1^2 p_2^{(n-2)/2} \right), & n \text{ even,} \end{cases} \quad (\text{dihedral: bracelets})$$

$$Z(\text{cube faces}) = \frac{1}{24} (p_1^6 + 3p_1^2 p_2^2 + 6p_1^2 p_4 + 6p_2^3 + 8p_3^2) \quad (\text{rotation group on 6 faces})$$

Face colourings of the cube in k colours: $\frac{1}{24} (k^6 + 3k^4 + 12k^3 + 8k^2)$, giving 10 for $k = 2$ and 57 for $k = 3$.

A.5 Generating Function Dictionary

Sequence / structure	Generating function	Type
$1, 1, 1, \dots$	$\frac{1}{1-x}$	OGF
$\binom{n+k-1}{k-1}$ (stars and bars)	$\frac{1}{(1-x)^k}$	OGF
Fibonacci F_n	$\frac{x}{1-x-x^2}$	OGF
Catalan C_n	$\frac{1-\sqrt{1-4x}}{2x}$	OGF
Partitions $p(n)$	$\prod_{k \geq 1} \frac{1}{1-x^k}$	OGF
Derangements D_n	$\frac{e^{-x}}{1-x}$	EGF
Bell numbers B_n	e^{e^x-1}	EGF
Surjections onto $[k]$	$(e^x - 1)^k$	EGF
Labelled trees on $[n]$ (n^{n-1} rooted)	$A(x) = x e^{A(x)}$	EGF

A.6 Answers and Hints to Selected Exercises

The exercises in this book were designed to be fought with, not looked up; hence this section supplies terminal numerical answers and directional hints only, for a selection of the computational problems, keyed by chapter. Where an exercise asked for a proof, no answer is given — the proof *is* the answer, and the chapter containing the exercise contains every tool it needs.

Ch. 2 Strings avoiding two consecutive ones, length 10: $F_{12} = 144$.

Ch. 3 Rank of RANDOM among its rearrangements: 614. Arrangements of MISSISSIPPI: 34650.

Ch. 4 Eight guests at a round table, two insisting on adjacency: $2 \cdot 6! = 1440$.

Ch. 5 $\sum_r \binom{n}{r}^2 = \binom{2n}{n}$ is Vandermonde with $m = n$, $r = n$, plus symmetry.

Ch. 6 Solutions of $a + b + c = 20$ in positive integers with $a \leq b \leq c$: count weak compositions, then quotient by symmetry with care at equalities: 33.

Ch. 7 Integers in $[1, 1000]$ coprime to 30: 266. Derangement of 5 letters: 44.

Ch. 8 Highest power of 5 dividing $100!$: 24 (Legendre). Parity of $\binom{100}{37}$: even, by Lucas in base 2.

Ch. 9 Standard Young tableaux of shape 3×3 : 42 (hook lengths). Paths avoiding the diagonal: reflect (Chapter 9, ballot argument).

Ch. 11 $\{7\}_3 = 301$; check against the surjection count $3! \cdot 301 = 1806 = \text{Sur}(7, 3)$.

Ch. 12 Coefficient extraction for bounded stars-and-bars, $x_1 + \dots + x_4 = 20$, $0 \leq x_i \leq 8$: 286.

Ch. 13 Domino tilings of 2×10 : $F_{11} = 89$.

Ch. 14 Lyndon words of length 6 over two letters: $\frac{1}{6}(2^6 - 2^3 - 2^2 + 2) = 9$.

Ch. 16 Necklaces, 6 beads, 2 colours: 14; with exactly three black: 4. Bracelets, 4 beads, 2 colours: 6.

Ch. 18 $p(9) = 30$ by the pentagonal recurrence; self-conjugate partitions of n = partitions of n into distinct odd parts.

Ch. 19 $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$; at $q = 2$ this is 35, the number of 2-dimensional subspaces of $\mathbb{F}_2^4$. Lines of $\mathbb{F}_3^3$ through the origin: 13.

Ch. 20 Sum-free subset of size $> n/3$: dilate by a uniformly random residue modulo a prime $p \equiv 2 \pmod{3}$ and average over the middle third.

A.7 Notation Index

Symbol	First used	Meaning
$[n]$	Ch. 1	the set $\{1, 2, \dots, n\}$
$ A $	Ch. 1	cardinality of the set A
$n^{\underline{r}}$	Ch. 3	falling factorial $n(n-1)\cdots(n-r+1)$
$x^{\overline{n}}$	Ch. 11	rising factorial $x(x+1)\cdots(x+n-1)$
$\binom{n}{r}$	Ch. 5	binomial coefficient
D_n	Ch. 7	number of derangements of $[n]$
C_n	Ch. 9	Catalan number
$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}, B_n$	Ch. 11	Stirling second kind; Bell number
$\begin{bmatrix} n \\ k \end{bmatrix}$	Ch. 11	(unsigned) Stirling first kind
$\varphi(n)$	Ch. 7	Euler totient
μ	Ch. 14	Möbius function (of a poset, or classical)
$\text{Orb}(x), \text{Stab}(x), \text{Fix}(g)$	Ch. 16	orbit, stabiliser, fixed-point set
$Z(G)$	Ch. 16	cycle index of the group G
$p(n)$	Ch. 18	number of integer partitions of n
$[n]_q, [n]_{q!}, \begin{bmatrix} n \\ r \end{bmatrix}_q$	Ch. 19	q -integer, q -factorial, Gaussian binomial
$\mathbb{F}_q$	Ch. 19	finite field with q elements